ELE6202E - Multivariable Systems

Lecture 1: Introduction

Fall 2026

Instructor: Bowen Yi

https://bwyi.github.io

POLYTECHNIQUE MONTRÉAL

Acknowledgments

These lecture slides have been prepared and are maintained by Bowen Yi. They build in part on earlier course materials developed at Polytechnique Montréal by Professors Jérôme Le Ny and Lahcen Saydy.

Primary References

The presentation also draws on several standard textbooks in linear systems theory, in particular:

- W. J. Rugh, *Linear System Theory*;
- F. M. Callier and C. A. Desoer, *Linear System Theory*;
- P. J. Antsaklis and A. N. Michel, *Linear Systems*.

Additional references are cited where appropriate. Any errors or omissions remain the responsibility of the instructor.

About the Instructor

- Dr. Bowen Yi
- Professeur adjoint in la section d'Automation et systèmes du Département de génie électrique
- Office: M-5418
- Email: bowen.yi@polymtl.ca
- Phone: +1 (514) 340 4711 ext. 3908
- Research interests:
  - Estimation, learning and control of nonlinear systems
  - Robotics

Contents

1. About the Course

Course Objectives

- This course teaches fundamental elements of the theory of linear dynamical systems. It covers standard material for a first year graduate course in control, preparing for more advanced topics in control theory, robotics, optimization and more.

- Students should be able to
  - Modelling: Use different representations to model linear multivariable systems (I/O models, transfer functions, state-space models)
  - Analysis: Apply appropriate analysis tools to different models, such as to determine stability, controllability, observability, minimality
  - Design: Transform into certain canonical forms, and design controllers and observers (separation principle, parameterization of all stabilizing controllers, optimization-based design)
  - Start reading the scientific literature on control theory.

Course Information

- Mandatory materials for the PhD qualifying exam for student in the Automation Section at Polytechnique.
- Website: Moodle
- Assignment solutions should be submitted through GradeScope.
- All course materials are presented in these slides. There is no official textbook, but you should refer to some textbooks on linear systems theory to have a better understanding.
- Prerequisites: ELE3201 Asservissements or equivalents

We assume that you are comfortable with matrix theory and calculus and can perform basic proofs and calculations.

Evaluation

- Homework: 6 graded homework assignments (5% each)

Submit individually; Not allowed to share the final answer or detailed methods. Violations will be reported to Polytechnique and handled under the academic integrity policy.

- Exams:
  - Midterm exam (30%, Week 6, October 7)
  - Final exam (40%)

- Grading: For late submission
  - 15% if late by less than 12h
  - -30% if late by more than 12h but less than 24h
  - 0 if late by more than 24h.

Other Information

- Two breaks each lecture (remind me if I forget).

- Post your any related questions on the Moodle forum rather than by emailing them to me directly.

- Check your email regularly – important announcements sent via emails.

- Generative AI tools are allowed: need to follow "*Utilisation des outils d'intelligence artificielle générative*" at Polytechnique.

Students who do not complete the assignments independently are unlikely to succeed in the exams.

Notes and Texts

There is no required textbook. Reading materials include:

- W. J. Rugh, *Linear System Theory*, 1996. $\star$
- F. M. Callier and C. A. Desoer, *Linear System Theory*. $\star$
- T. Kailath, *Linear Systems*, Prentice–Hall, 1980.
- P. J. Antsaklis and A. N. Michel, *Linear Systems*, 1997. (Online textbook available to students at Polytechnique)
- Lecture notes for the course 6.241J (Dynamic Systems and Control) at MIT, by M. Dahleh, M. Dahleh and G. Verghese (Available online)
- J. P. Hespanha, *Linear Systems Theory*, 2018. (Textbook sold at the COOP)
- Chen, Chi-Tsong, *Linear System Theory And Design*, 3rd Ed., 1999. (Available online)

2. Background

Why study multivariable dynamical systems?

Applications are ubiquitous:

- Robots (medicine, soft, manipulator)

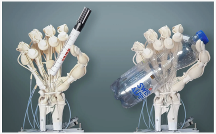


Figure 1: 3D-printed a robotic hand (ETH Zurich)

- Artificial intelligence: recurrent neural network (RNN)

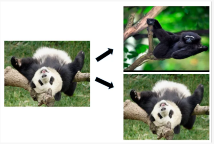
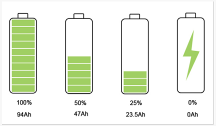
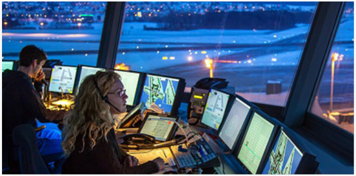


Figure 2: Adversarial examples (trick image classifiers)

- Chemical process control: state-of-charge in batteries



- Air traffic control



- Autopilot on aircraft or spacecraft
- Mechatronics: control of electromechanical devices, MEMS
- Power systems/electronics: generators, sensorless control of motors, power networks
- Quantum physics: quantum computation
- Genetic circuit: control of evolution of gene
- $\cdots$

Origins and Brief History

- Key concept in automatic control is feedback, widely existing in nature and industry
  1. Body temperature regulation (metabolic + evaporative)
  2. Body movements (optical feedback)
  3. Perhaps the most famous example: Centrifugal steam engine governor (James Watt)

Missing image: Lecture˙1/figure5.png

- Limits of "classical" theory: linear time-invariant (LTI), single-input-single-output (SISO), analysis rather than synthesis (design by trial and error)

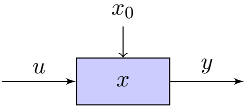
Origins and Brief History

- Parts of Linear Dynamic System theory can be tracked back to the 19th century.
- Builds upon classical circuits & systems (1920s, transfer functions . . .), but with more emphasis on linear algebra
- First engineering application: aerospace (1960s)
- Transitioned to ubiquitous in 1980s
- Additional Notes on *Brief History of Feedback Control*: `https://lewisgroup.uta.edu/history.htm`.

3. Models and Modelling

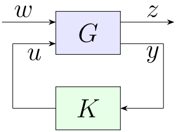
Models

- Mathematical models include
  - input $u \in \mathbb{R}^m$
  - output $y \in \mathbb{R}^p$
  - internal state $x \in \mathbb{R}^n$ (that may or may not be fully measured)
  - initial condition $x_0 := x(0)$



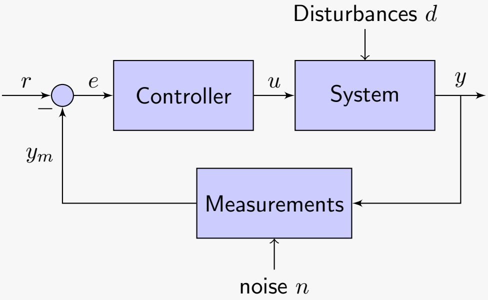
u, y, x are variables of time, e.g. $u(t)$ or $u(k)$.

- A general view of control: measure output y and manipulate input u, to achieve desired response from w to z.
  - G: generalized plant
  - K: controller to design
  - Example: Follow a reference signal ω; attenuate disturbance ω.



Models (cont'd)

Classical Control Loop



Even if all signals are scalar, we get an MIMO generalized plant with

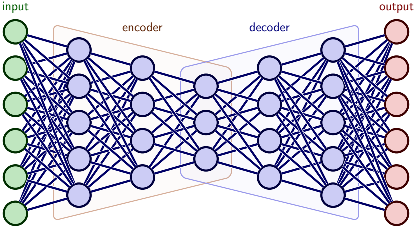
$$\omega = \begin{bmatrix} r \\ d \\ n \end{bmatrix}.$$

Modeling

- Key points for a model:
  1. Compact & Abstract
  2. Powerful for prediction, analysis, and design
- Different types of models:
  1. Rule-based model:

$$\begin{gathered}\text{if } A_1, \text{ then } B_1 \\ \vdots \\ \text{if } A_k, \text{ then } B_k\end{gathered}$$

  2. Neural networks



Modeling (cont'd)

3. Bode plot (in classical control engineering)

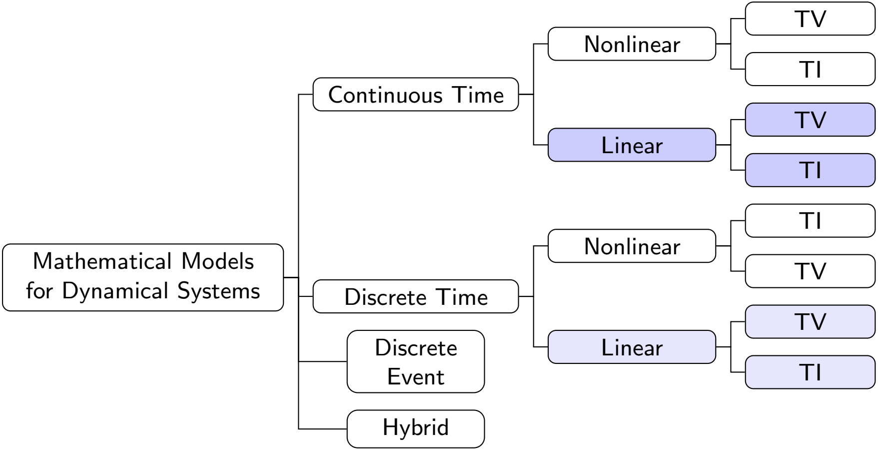
Missing image: Lecture˙1/bode.png

4. Mathematical Models
   - Continuous-time linear dynamical system (ordinary diff eqs, ODE)

$$\frac{dx}{dt} = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t).$$

   - ...

Modeling (cont'd)



† TV: Time-varying, TI: Time invariant, ☐ Focus of this course.

Nomenclature

Notations: $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$, and

$$
\begin{aligned}
\mathbb{R}_+ &= \{x \in \mathbb{R} : x \geq 0\}, & \mathbb{R}_{>0} &= \{x \in \mathbb{R} : x > 0\} \\
\mathbb{R}_- &= \{x \in \mathbb{R} : x \leq 0\}, & \mathbb{R}_{<0} &= \{x \in \mathbb{R} : x < 0\} \\
\mathbb{C}_+ &= \{x \in \mathbb{C} : \mathrm{Re}(x) \geq 0\}, & \mathbb{C}_{>0} &= \{x \in \mathbb{C} : \mathrm{Re}(x) > 0\} \\
\mathbb{C}_- &= \{x \in \mathbb{C} : \mathrm{Re}(x) \leq 0\}, & \mathbb{C}_{<0} &= \{x \in \mathbb{C} : \mathrm{Re}(x) < 0\}
\end{aligned}
$$

Quantifiers:

$\in$	in	$\exists$	there exists
$\forall$	for all in	$\exists!$	there exists a unique
s.t.	such that	$\equiv$	always equal to
$\implies$	implies	$\iff$	if and only if (iff)

Algebraic Concepts and Applications

Algebraic concepts	**Examples of usage in linear systems**
linear (vector) space	state space, input space, output space
linear map	reachability map and observability map
normed space	stability analysis
inner product, adjoint	controllability, observability Gramians

Types of Mathematical Models

M1 CT: Ordinary differential equations (ODEs):

$$F\left(y^{(n)}(t), \ldots, y(t), u^{(m)}(t), \ldots, u(t), t\right) = 0, \quad \text{IC at } t_0. \tag{1}$$

N.B.: $n \geq m$ for causality $u \to y$. IC: Initial condition.

M2 Linear constant coefficients:

$$a_n y^n + \ldots, a_0 y = b_m u^m + \ldots + b_0 u. \tag{2}$$

a_i, b_j may be matrices for MIMO systems.

Types of Mathematical Models (cont'd)

M3 DT: Difference equations:

$$y[k+n] + a_1[k]y[k+n-1] + \ldots + a_n[k]y[k] \\ = b_0[k]u[k] + \ldots + b_m[k]u[k+m]$$

N.B.: $n \geq m$ for causality $u \to y$.

M4 Infinite dimensional systems (Partial differential equations, PDEs):

$$\frac{\partial^2 y}{\partial t^2}(s,t) = c^2 \frac{\partial^2 y}{\partial s^2}(s,t) \tag{3}$$

Dynamics of wave equations with the boundary conditions forced by u, e.g. $y(c,t) = u(t)$ and the IC $y(\cdot, 0) = y_0(\cdot)$.

Types of Mathematical Models (cont'd)

M5 Polynomial matrix description: Denote the differential operator $p := \frac{d}{dt}$. Model M2 can be rewritten as

$$D(p)[y] = N(p)[u] \tag{4}$$

with polynomial matrices $D(p) = a_n p^n + \ldots, a_0$ and $N(p) = b_m p^m + \ldots b_0$.
More general (naturally in interconnected systems):

$$P(p)[z] = Q(p)[u], \quad y = R(p)[z] + S(p)[u]. \tag{5}$$

M6 Transfer functions and matrices: For M2 (LTI systems), assuming zero IC, we take the Laplace transforms

$$D(s)Y(s) = N(s)U(s) \quad \Longrightarrow \quad \text{TF: } H(s) = D(s)^{-1}N(s)$$

The matrices contain rational functions of s.

Types of Mathematical Models (cont'd)

M7 Input-output operator: G defines an operator from $u(t)$ to $y(t)$, i.e.

$$y(t) = G[u](t). \tag{6}$$

Example: revolution operator $y(t) = \int_{-\infty}^{t} g(t, -\tau)u(\tau)d\tau$.

M8 State-space models: Generality & Good for computations (numerical codes)

> First-order ODE (CT) or difference equations (DT):
>
> $$\begin{aligned} \dot{x} &= f(x, u, t), \quad x(0) = x_0, \\ y &= h(x, u, t) \end{aligned} \tag{7}$$

Minimal dimension of states: memory & complexity (behaviour + control)

Example 1: A Rocket with Time-Varying Mass

Suppose a rocket moves vertically upward. Let

$h(t)$: altitude, $v(t)$: vertical velocity, $m(t)$: rocket mass.

Assume:

- gravitational acceleration is the constant g,
- the exhaust velocity v_e is constant,
- the mass variation rate is constant: $\dot{m}(t) = u_0$,
- $m(0) = m_0$, $h(0) = 0$, $v(0) = 0$.

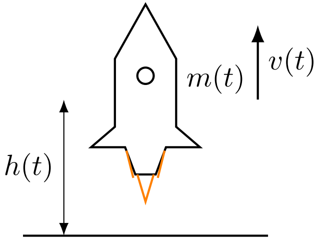
Newton's law yields

$$m(t)\dot{v}(t) = -m(t)g + v_e u_0.$$

Also,

$$\dot{h}(t) = v(t), \qquad \dot{m}(t) = u_0.$$

Hence, $m(t) = m_0 + u_0 t$.



State-Space Model

Choose the state variables $x := [h \quad v]^\top$ and let the output be the altitude: $y = x_1$.
Since

$$\dot{h}(t) = v(t), \qquad \dot{v}(t) = -g + \frac{v_e u_0}{m_0 + u_0 t},$$

the system can be written as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -g + \dfrac{v_e u_0}{m_0 + u_0 t} \end{bmatrix}, \qquad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

This is a linear state equation with a *time-varying forcing term*. It is not a time-varying (a concept to learn later) linear system in the matrices $A(t), B(t)$; instead, the time dependence appears in the forcing term through the mass $m(t) = m_0 + u_0 t$.

Example 2: Satellite Motion

Consider a unit-mass satellite moving in a plane. In polar coordinates,

$$r(t) : \text{radial distance}, \qquad \theta(t) : \text{angular position}.$$

The control inputs are the radial and tangential thrusts $u = [u_r \;\; u_\theta]^\top$. The equations of motion are

$$\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_r,$$

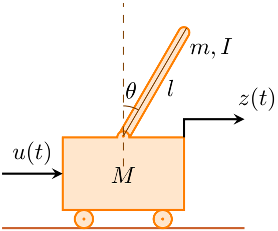
$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \frac{u_\theta}{r}.$$

Selecting $x = [r, \dot{r}, \theta, \dot{\theta}]^\top$ yields the state-space model:

$$\dot{x} = \begin{bmatrix} x_2 \\ x_1 x_4^2 - \frac{k}{x_1^2} + u_1 \\ x_4 \\ -\frac{2x_2x_4}{x_1} + \frac{u_2}{x_1}. \end{bmatrix} \tag{8}$$

Example 3: Inverted Pendulum on a Cart

- Measurements (outputs): $y = \begin{bmatrix} z \\ \theta \end{bmatrix}$
- Action on wheels (control input): $u \in \mathbb{R}$
- Target: Requires a controller for stabilization in the upward position.
- Euler-Lagrange equations (2nd order ODE):



$$
\begin{aligned}
(M+m)\ddot{z} + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta) &= u \\
m\ddot{z}\cos\theta + ml\ddot{\theta} &= mg\sin\theta.
\end{aligned}
$$

From the ODEs to a State-Space Model

First, collect the acceleration terms:

$$\begin{bmatrix} M+m & ml\cos\theta \\ m\cos\theta & ml \end{bmatrix} \begin{bmatrix} \ddot{z} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u + ml\dot{\theta}^2\sin\theta \\ mg\sin\theta \end{bmatrix}.$$

Since

$$\det \mathcal{M}(\theta) = ml\big(M + m - m\cos^2\theta\big) > 0,$$

we can solve for $\ddot{z}$ and $\ddot{\theta}$.

Selecting the state

$$x := \begin{bmatrix} z & \dot{z} & \theta & \dot{\theta} \end{bmatrix}^\top,$$

one has

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \ddot{z}, \qquad \dot{x}_3 = x_4, \qquad \dot{x}_4 = \ddot{\theta}.$$

We now have the state-space model:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{mlx_4^2 \sin(x_3) - mg \sin(x_3) \cos(x_3) + u}{(M+m) - m\cos^2(x_3)} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{(M+m)g \sin(x_3) - mlx_4^2 \sin(x_3) \cos(x_3) - u\cos(x_3)}{l(M+m-m\cos^2(x_3))} \\
y_1 &= x_1 \\
y_2 &= x_3.
\end{aligned}$$

It will be useful for the control design process to keep this system around a desired equilibrium $x = 0$.

4. Basic Properties of Systems

Memory & Memoryless

Definition 1 (Memoryless) A system is memoryless if the output $y(t)$ at each t depends on the input only at the same instant, i.e.

$$y(t) = f(t, u(t)).$$

- Note: Possible futures are independent of the past. No state is needed to predict its evolution.
- Example: Resistor

$$y(t) = \mathbf{v}(t) = R\mathbf{i}(t) \tag{9}$$

- Not an example: Capacitor

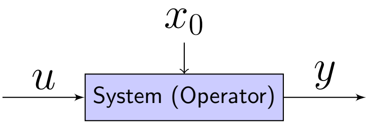
$$\dot{\mathbf{v}} = \frac{1}{C}\mathbf{i} \quad \Longrightarrow \quad \mathbf{v}(t) = \mathbf{v}(0) + \frac{1}{C}\int_0^t \mathbf{i}(s)ds,\ t \geq 0. \tag{10}$$

Linearity

A system can be viewed as an operator

$$G : \mathcal{U} \times \mathcal{X} \times \mathcal{T} \to \mathcal{Y}$$

transforming an input signal $u \in \mathcal{U}$ and initial condition $x_0 \in \mathcal{X}$ at $t \in \mathcal{T}$ to output signal $y \in \mathcal{Y}$, in which $\mathcal{T}$ is the time axis (e.g. $\mathbb{R}, \mathbb{N}, [0, T], \cdots$).



Linearity (cont'd)

Definition 2 (Linearity) An input-output system $G : \mathcal{U} \times \mathcal{X} \times \mathcal{T} \to \mathcal{Y}$ is linear if $\mathcal{U}, \mathcal{Y}, \mathcal{X}$ are vector spaces, and $G(\cdot, \cdot; t_0)$ is linear in (u, x_0) for each $t_0 \in \mathcal{T}$, i.e.

- $G(u, x_0; t_0) = G_{zi}(x_0; t_0) + G_{zs}(u; t_0)$
- $G_{zi}, G_{z,s}$ are linear:

$$\begin{aligned} G_{zi}(\alpha x_{0,1} + \beta x_{0,2}; t_0) &= \alpha G_{zi}(x_{0,1}; t_0) + \beta G_{zi}(x_{0,2}; t_0) \\ G_{zs}(\alpha u_1 + \beta u_2; t_0) &= \alpha G_{zs}(u_1; t_0) + \beta G_{zs}(u_2; t_0). \end{aligned}$$

Remarks:

- $G_{zi}(\cdot; t_0) \in \mathcal{Y}$ is the zero input response/homogeneous response.
- $G_{zs}(\cdot; t_0) \in \mathcal{Y}$ is the zero state response/zero initial condition response/forced response.
- Implications: $G_{zi}(0, t_0) = 0, \quad G_{zs}(0, t_0) = 0.$

Causality

In this course, we are mostly interested in causal systems: the output at any time $t \in \mathcal{T}$ depends only on the values of the past input on $(-\infty, t]$.

Definition 3 (Causality) A system $G : \mathcal{U} \to \mathcal{Y}$ is said to be causal if

$$P_T G P_T u = P_T G u, \quad \forall T, \, \forall u \in \mathcal{U}, \tag{11}$$

where the truncation operator P is defined as

$$(P_T u)(t) = \begin{cases} u(t) & \text{for } t \leq T \\ 0 & \text{for } t > T. \end{cases}$$

A state-space model is always causal.

- Roughly, a system is causal if its output at any time t depends only on values of the input evaluated for $t \leq s$. Thus $y(t)$ depends only on $u(s)$, $s \leq t$.
- Example: The discrete-time system

$$y(n) = u(n) - u(n+1), \quad n \in \mathbb{Z}$$

is *not* causal.
- Noncausal systems are useful in offline data processing, where future data are already available. E.g., the centered moving-average filter

$$y(n) = \frac{1}{2M+1} \sum_{k=-M}^{M} u(n-k)$$

smooths data using both past and future samples.

Time-Invariance

Definition 4 (Time-invariance) A system $G : \mathcal{U} \times \mathcal{X} \times \mathcal{T} \to \mathcal{Y}$ is said to be time invariant if

$$G(S_T u, x_0; t_0 + T) = S_T G(u, x_0; t_0) \tag{12}$$

in which the shifting operator is defined as $S_T u(t) = u(t - T)$.

- Intuitively, shifting input by τ shifts output by the same amount.
- Time-invariant nonlinear systems

$$\dot{x} = f(x, u), \quad y = h(x, u) \tag{13}$$

  with f and h independent of t.
- Autonomous systems: $f(x), h(x)$ only depend on x.
- Example 1: A Rocket with Time-Varying Mass ?

5. Linear State-Space Model and Linearization

Linear State-Space Model

Continuous-time State-Space Models

$$\begin{aligned} \frac{d}{dt}x(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{14}$$

Discrete-time State-Space Models

$$\begin{aligned} x[k+1] &= A[k]x[k] + B[k]u[k] \\ y[k] &= C[k]x[k] + D[k]u[k]. \end{aligned} \tag{15}$$

- In the above, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$.
- LTI:

$$\begin{aligned} \dot{x} &= Ax + Bu \quad \text{or} \quad x[k+1] = Ax[k] + Bu[k] \\ y &= Cx + Du \end{aligned}$$

Equilibrium

Most, if not all, systems are nonlinear, but linear models are easier to handle and naturally arise when studying a nonlinear system around an equilibrium.

Definition 5 (Equilibrium) Consider a nonlinear state space model:

$$\dot{x} = f(x, u, t), \quad y = h(x, u, t) \tag{16}$$

with $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$. For a given constant $u_\star$, we call constant $x_\star \in \mathbb{R}^n$ an equilibrium if

$$f(x_\star, u_\star, t) = 0, \quad \forall t. \tag{17}$$

- Denote the solution from x_0 at t_0 as $\tilde{x}(t; x_0, t_0)$. We have

$$x_\star \ \text{ is an equilibrium.} \quad \Longrightarrow \quad \tilde{x}(t; x_\star, t_0) \equiv x_\star$$

- We are interested in the dynamical behaviour around $(x_\star, u_\star)$.

Linearization

Consider the trajectory near $(x_\star, u_\star)$, i.e.

$$x = x_\star + \delta x, \quad u = u_\star + \delta u$$

with small amounts $(\delta x, \delta u)$. Performing a Talyor expansion yields

$$\begin{aligned}
\cancel{\dot{x}_\star} + \delta\dot{x} &= f(x_\star + \delta x, u_\star + \delta u, t) \\
&= \cancel{f(x_\star, u_\star, t)} + \underbrace{\frac{\partial f}{\partial x}(x_\star, u_\star, t)}_{A(t)}\delta x + \underbrace{\frac{\partial f}{\partial u}(x_\star, u_\star, t)}_{B(t)}\delta u + \text{H.O.T.}
\end{aligned}$$

$$\begin{aligned}
y_\star + \delta y &= h(x_\star + \delta x, u_\star + \delta u, t) \\
&= h(x_\star, u_\star, t) + \underbrace{\frac{\partial h}{\partial x}(x_\star, u_\star, t)}_{C(t)}\delta x + \underbrace{\frac{\partial h}{\partial x}(x_\star, u_\star, t)}_{D(t)}\delta u + \text{H.O.T.}
\end{aligned}$$

in which we have defined the steady-state output $y_\star = h(x_\star, u_\star, t)$.

Linearized Model

$$\begin{aligned} \delta \dot{x} &= A(t)\delta x + B(t)\delta u \\ \delta y &= C(t)\delta x + D(t)\delta u. \end{aligned} \tag{18}$$

with

$$\begin{aligned} A(t) &= \left.\frac{\partial f}{\partial x}\right|_{(x_\star, u_\star, t)}, \quad B(t) = \left.\frac{\partial f}{\partial u}\right|_{(x_\star, u_\star, t)} \\ C(t) &= \left.\frac{\partial h}{\partial x}\right|_{(x_\star, u_\star, t)}, \quad D(t) = \left.\frac{\partial h}{\partial u}\right|_{(x_\star, u_\star, t)}. \end{aligned}$$

- Only true as long as $(\delta u, \delta x, \delta y)$ are small.
- Can be done along a non-equilibrium trajectory $(x(t), u(t))$ – a time-invariant system would yield a *time-varying* linearized model.
- If nominal solution is periodic, then linearized system is also periodic.

Example 2 (cont'd): Linearization about a Circular Orbit

With zero thrust, consider the circular orbit

$$x_\star(t) = \begin{bmatrix} r_0 \\ 0 \\ \omega t + \theta_0 \\ \omega \end{bmatrix}, \qquad u_\star = 0.$$

The radial equation requires

$$0 = r_0\omega^2 - \frac{k}{r_0^2}, \qquad \Longrightarrow \qquad k = r_0^3\omega^2.$$

- Linearizing around $(x_\star, u_\star)$ gives

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2r_0\omega \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{2\omega}{r_0} & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{r_0} \end{bmatrix} \delta u.$$

- If

$$y = \begin{bmatrix} r & \theta \end{bmatrix}^\top,$$

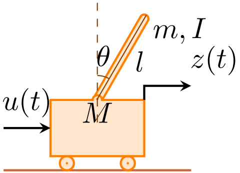
then

$$\delta y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \delta x.$$

Although the nominal solution is time varying, the linearized model is LTI because the dynamics are independent of θ.

Exercise: Example 3 – Inverted Pendulum on a Cart

Linearize at $x \equiv 0,\ u \equiv 0$:



$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{mlx_4^2 \sin(x_3) - mg \sin(x_3)\cos(x_3) + u}{(M+m) - m\cos^2(x_3)}$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{(M+m)g\sin(x_3) - mlx_4^2\sin(x_3)\cos(x_3) - u\cos(x_3)}{l\big((M+m) - m\cos^2(x_3)\big)}$$

$$y_1 = x_1$$

$$y_2 = x_3$$

Solution

Around $(x_\star, u_\star) = (0, 0)$, we have

$$\delta\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(M+m)g}{Ml} & 0 \end{bmatrix}}_{A} \delta x + \underbrace{\begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{Ml} \end{bmatrix}}_{B} \delta u,$$

$$\delta y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \delta x, \qquad D = 0.$$

Summary

- Understand the course objectives
- Model and its classification
- Key properties of a model (memory, linearity, causality, equilibrium)
- State-space modeling
- Linearization of a nonlinear model

Sources and Further Reading

- System models and representations: The discussion of differential equations, input–output operators, transfer matrices, and state-space models draws primarily on Rugh, Callier and Desoer, and Antsaklis and Michel.
- Basic system properties: The treatments of memory, linearity, causality, and time invariance follow (Antsaklis and Michel).
- State-space modelling and linearization: The presentation of equilibria, perturbation models, and local linearization draws in part on Khalil.[1]
- Historical perspective: The brief discussion of feedback and the development of automatic control is informed by Bennett's historical survey (Bennett, *IEEE Control Systems Magazine*, 1996).

The rocket, satellite, and inverted-pendulum examples were prepared by the instructor from standard models in mechanics and control.

[1] H. K. Khalil, *Nonlinear Systems*, 3rd ed., Prentice Hall, 2002.