ELE6202E - Multivariable Systems

Lecture 3: Solutions to Linear State-Space Models

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POLYTECHNIQUE MONTRÉAL

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1. Existence and Uniqueness of the Solution of an ODE

Existence and Uniqueness of the Solution of an ODE

- Consider the basic differential equation

$$\dot{x} = f(x,t), \quad x(t_0) = x_0$$

  where $x \in \mathbb{R}^n$, for $t \geq 0$, and $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$.

- Let $\mathcal{I}$ be a set in $\mathbb{R}_+$ which contains at most a finite number of points per unit interval. Here $\mathcal{I}$ is the set of possible discontinuous points: *it may be empty*.

Assumption 1 *Suppose that the function $f(\cdot,\cdot)$ satisfies the following two assumptions:*

1. *For each fixed $x \in \mathbb{R}^n$, the function $f(x,\cdot)$ is Piecewise Continuous in t, i.e. $f(x,\cdot) : \mathbb{R}_+ \backslash \mathcal{I} \to \mathbb{R}^n$ is continuous, and for any $\tau \in \mathcal{I}$, $\lim_{t\to\tau^-} f(x,t)$ and $\lim_{t\to\tau^+} f(x,t)$ exist;*

2. *The function $f(\cdot,\cdot)$ is Lipschitz continuous in x, i.e., there exists a piecewise continuous function $k(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|f(x_1,t) - f(x_2,t)\| \le k(t)\|x_1 - x_2\|$$

for all $t \in \mathbb{R}_+$ and all $x_1, x_2 \in \mathbb{R}^n$.

Existence and Uniqueness Theorem

Theorem 1 *Under the above assumption, we have*[a]

1. *For each* $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, *there exists a continuous function* $\phi : \mathbb{R}_+ \to \mathbb{R}^n$ *such that*

$$\begin{aligned} \dot{\phi}(t) &= f(\phi(t), t), \quad \forall t \in \mathbb{R}_+ \backslash \mathcal{I} \\ \phi(t_0) &= x_0; \end{aligned} \tag{1}$$

2. *This function is unique.*

The function ϕ *is called the solution through* (t_0, x_0) *of the ODE.*

[a] Its proof can be found in (Sastry, Nonlinear Systems: Analysis, Stability and Control, page 86).

Question: How about discrete-time systems?

Exercises

1. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \sin(x_1) \\ x + 2\cos(x_2) \end{bmatrix}$.
2. $\dot{x} = x^2$.

Linear Systems

The linear system

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned}$$

with proper dimensions, satisfies the conditions of the Existence and Uniqueness Theorem for ODEs.

$$\begin{aligned} \|f(x_1, t) - f(x_2, t)\| &= \|A(t)(x_1 - x_2)\| \\ &\leq \|A(t)\| \|x_1 - x_2\|. \end{aligned}$$

2. Solution to Discrete-Time Systems

Solutions to Homogeneous/Unforced Systems (DT)

Theorem 2 *The homogeneous system*

$$x_{k+1} = A_k x_k, \quad x_{k_0} = x_0 \tag{2}$$

with $x_k := x[k] \in \mathbb{R}^n$*, has a unique solution for* $k \geq k_0$*:*

$$x_k = \Phi(k, k_0) x_0 \tag{3}$$

with the state-transition matrix

$$\Phi(k, k_0) = \begin{cases} I_n & \text{for } k = k_0 \\ A_{k-1} \dots A_{k_0} & \text{for } k > k_0. \end{cases}$$

This solution is for $k \geq k_0$, but not $k < k_0$. Why?

Sketch of Proof

We observe that

$$
\begin{aligned}
x_{k+2} &= A_{k+1}A_k x_k \\
&\vdots \\
x_n &= A_{n-1}A_{n-2}\ldots A_k x_k = \prod_{j=k}^{n-1} A_j x_k.
\end{aligned}
\tag{4}
$$

This suggests that the state transition matrix $\Phi(n,k)$ is given by

$$
\begin{aligned}
\Phi(n,k) &= \prod_{j=k}^{n-1} A_j, \quad n > k, \\
\Phi(k,k) &= I.
\end{aligned}
$$

Applying the above to the initial-value problem, we complete the proof. ■

Properties of STM (Discrete-time). The state transition matrix $\Phi(n, k)$ satisfies

1. For $k_1 \leq k_2 \leq k_3$,

$$\Phi(k_3, k_2)\Phi(k_2, k_1) = \Phi(k_3, k_1).$$

2. The i-th column of $\Phi(k, k_0)$ is solution of the system

$$x_{k+1} = A_k x_k, \quad x_{k_0} = e_i := \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

Hint: Φ is the solution of $\Phi(k+1, k_0) = A_k \Phi(k, k_0)$, $\Phi(k_0, k_0) = I$.

Solutions to Nonhomogeneous/Forced Systems (DT)

Theorem 3 *The forced system*

$$x_{k+1} = A_k x_k + B_k u_k, \quad x_{k_0} = x_0 \tag{5}$$

has a unique solution for $k \geq k_0$:

$$x_k = \underbrace{\Phi(k, k_0)x_0}_{\text{zero-input response}} + \underbrace{\sum_{j=k_0}^{k-1} \Phi(k, j+1)B_j u_j}_{\text{zero-state response}} \tag{6}$$

Proof

By induction, at $k = k_0$, we have

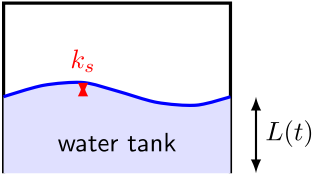
$$x_{k_0} = \Phi(k_0, k_0)x_0 + 0 = x_0.$$

If it is true for k, for $k + 1$

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ &= A_k \left(\Phi(k, k_0)x_0 + \sum_{j=k_0}^{k-1} \Phi(k, j+1)B_j u_j \right) + B_k u_k \\ &= \Phi(k+1, k_0)x_0 + \sum_{j=k_0}^{k-1} \Phi(k+1, j+1)B_j u_j + \Phi(k+1, k+1)B_k u_k \\ &= \Phi(k+1, k_0)x_0 + \sum_{j=k_0}^{k} \Phi(k+1, j+1)B_j u_j, \end{aligned}$$

which verifies the form. ◁

Example 1: Sloshing Water Level



$L(t) = k_s \sin(\omega t + \phi)$ is the water level and k_s is the magnitude of the sinusoidal component.

Assume the known phase $\phi = 0$ and define

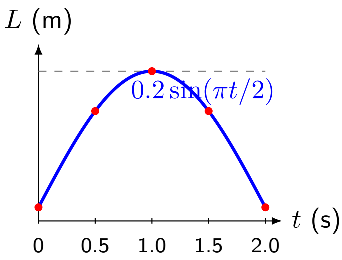
$$x(t) = \begin{bmatrix} L(t) \\ k_s \end{bmatrix}.$$

Exact sampling of the source model at times t_k gives

$$x_{k+1} = A_k x_k, \qquad A_k = \begin{bmatrix} 1 & \delta_k \\ 0 & 1 \end{bmatrix}$$

$$\delta_k := \sin(\omega t_{k+1}) - \sin(\omega t_k).$$

Reference: G. Welch, "Dynamic and Measurement Models," UNC-Chapel Hill COMP 145, 2001, Sections 1.1.3 and 1.2.3, pp. 1, 3–4. Source PDF.



The state-transition matrix telescopes:

$$\Phi(n,k) = A_{n-1}\cdots A_k = \begin{bmatrix} 1 & \sum_{j=k}^{n-1}\delta_j \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \sin(\omega t_n) - \sin(\omega t_k) \\ 0 & 1 \end{bmatrix}.$$

$\omega = \pi/2$ rad/s, $t_1 = 0.5, t_2 = 1$, $x_0 = [0, 0.2]^\top$, $\delta_0 = 0.707, \delta_1 = 0.293.$

First two samples:

$$x_1 = A_0 x_0 = \begin{bmatrix} 0.141 \\ 0.2 \end{bmatrix}, \quad x_2 = A_1 A_0 x_0 = \begin{bmatrix} 0.200 \\ 0.2 \end{bmatrix}.$$

3. Solution to Continuous-Time Systems

Solutions to Continuous-time Systems

Similar to DT systems but more technical.

Homogeneous/Unforced Systems

Consider the continuous-time homogeneous system

$$\dot{x} = A(t)x, \quad x(t_0) = x_0. \tag{7}$$

The $n \times n$ function $A(t)$ is continuous (i.e., each element $t \mapsto a_{ij}(t)$) and defined for all $t \geq t_0$.

Theorem 4 *(Uniqueness) Let $t \mapsto A(t)$ be continuous over $t \in \mathcal{T}$ ($t_0 \in \mathcal{T}$). Then, the system* (7) *has at most one solution $x(t)$ over $\mathcal{T}$, for a given $x(t_0) = x_0$.*

Proof

Assume that there are two distinct solutions x_1, x_2 from the same initial condition, and define $z = x_1 - x_2$. Then,

$$\dot{z} = A(t)z, \quad z(t_0) = 0. \tag{8}$$

Integrating the above equation, we have

$$\|z(t)\| = \left\| \int_{t_0}^{t} A(s)z(s)ds \right\| \leq 0 + \int_{t_0}^{t} \|A(s)\| \|z(s)\| ds.$$

Applying the Gronwall-Bellman ineq., we have $\|z(t)\| = 0$, $\forall t \geq t_0$. ■

Lemma 5 *(Gronwall-Bellman inequality)* Suppose that $\phi(t)$ and $v(t) \geq 0$ are continuous functions defined for $t \geq t_0$, and β is a constant. Then

$$\phi(t) \leq \beta + \int_{t_0}^{t} v(s)\phi(s)ds, \ t \geq t_0 \implies \phi(t) \leq \beta e^{\int_{t_0}^{t} v(s)ds}, \ t \geq t_0.$$

Proof of Gronwall-Bellman inequality ($\star$)

Define $r := \beta + \int_{t_0}^t v(s)\phi(s)ds$. Then

$$\dot{r} = v(t)\phi(t).$$

From $\phi(t) \leq \beta + \int_{t_0}^t v(s)\phi(s)ds,\ t \geq t_0$ and the assumption $v(t) \geq 0$,

$$\dot{r} = v(t)\phi(t) \leq v(t)r(t).$$

Multiply both sides by the positive function $\exp(-\int_{t_0}^t v(s)ds)$ to obtain

$$\frac{d}{dt}\left[r(t)e^{-\int_{t_0}^t v(s)ds}\right] \leq 0, \quad t \geq t_0. \tag{9}$$

Integrating both sides from t_0 to any $t \geq t_0$ gives

$$r(t)e^{-\int_{t_0}^t v(s)ds} - \beta \leq 0, \quad t \geq t_0,$$

and this yields RHS. ■

Theorem 6 *(Existence) Assume $t \mapsto A(t)$ is continuous on a compact interval $\mathcal{T} := [t_0, t_0+T]$. Then, the system has a (unique) solution on $\mathcal{T}$. Moreover, this solution can be written explicitly using the Peano-Baker series as*

$$x(t) = S_\infty(t)x_0,$$

where the series is defined as $S_k = \sum_{j=0}^{k} \Delta_j(t)$ with

$$\Delta_0(t) = I, \quad \Delta_1(t) = \int_{t_0}^{t} A(s)ds,$$

$$\Delta_2(t) = \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2)ds_2 ds_1,$$

$$\Delta_k(t) = \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2) \ldots \int_{t_0}^{s_{k-1}} A(s_k)ds_k \ldots ds_1,\ k \geq 1,$$

and S_k is uniformly absolutely convergent and the limit S_∞ is continuously differentiable.

Proof

Basic idea: Given t_0, x_0 and arbitrary time $T > 0$, we will construct a sequence of "approximate" solutions and prove their uniform convergence on $[t_0, t_0 + T]$.

The sequence of approximating functions on $[t_0, t_0 + T]$ is defined as:

$$x^0(t) = x_0$$

$$x^1(t) = x_0 + \int_{t_0}^{t} A(s)x^0(s)ds$$

$$\vdots$$

$$x^k(t) = x_0 + \int_{t_0}^{t} A(s)x^{k-1}(s)ds.$$

It can be written compactly using the Peano-Baker series:

$$x^k(t) = \sum_{j=0}^{k} \Delta_k(t)x_0 = S_k(t)x_0 = x^0(t) + \sum_{j=0}^{k-1}[x^{j+1}(t) - x^j(t)].$$

[Convergence] Let $\alpha := \max_{t \in \mathcal{T}} \|A(t)\|$ and $\beta := \int_{t_0}^{t_0+T} \|A(s)x_0\| ds$, which are finite. Then,

$$\|x^1(t) - x^0(t)\| = \left\| \int_{t_0}^{t} A(s)x_0 ds \right\| \leq \int_{t_0}^{t} \|A(s)x_0\| \, ds \leq \beta.$$

Consequently, it yields

$$\|x^2(t) - x^1(t)\| \leq \int_{t_0}^{t} \alpha\beta ds = \beta\alpha(t - t_0),$$

and

$$\|x^{j+1}(t) - x^j(t)\| \leq \int_{t_0}^{t} \|A(s)\| \|x^j(s) - x^{j-1}(s)\| ds \leq \beta \frac{\alpha^j (t - t_0)^j}{j!}$$

Using the Weierstrass M-Test, we can prove the convergence of the infinite series uniformly and absolutely on $\mathcal{I}$.

[Solution to the ODE] Taking $k \to \infty$, the limit of the sequence can be written as

$$x^{\infty}(t) = x_0 + \int_{t_0}^{t} A(s)x_0 ds + \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2)x_0 ds_2 ds_1 + \dots$$

Calculating the time derivative of the RHS:

$$\begin{aligned}
\dot{x}^{\infty}(t) &= 0 + A(t)x_0 + A(t) \int_{t_0}^{t} A(s)x_0 ds \\
&\quad + A(t) \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2)x_0 ds_2 ds_1 + \dots \\
&= A(t) \left[x_0 + \int_{t_0}^{t} A(s)x_0 ds + \int_{t_0}^{t} A(s_1) \int_{t_0}^{s_1} A(s_2)x_0 ds_2 ds_1 + \dots \right] \\
&= A(t)x^{\infty}(t).
\end{aligned}$$

Therefore, the infinite series expression is a solution to the homogeneous system. ■

Remarks

- In contrast to the DT case, we can have $t < t_0$ that allows computing backwards in time.
- The Weierstrass M-test is a test for determining whether an infinite series of functions converges uniformly and absolutely.

Weierstrass M-test. Suppose that (f_n) is a sequence of real- or complex-valued functions defined on a set A, and that $\exists$ a sequence of non-negative numbers (M_n) satisfying the conditions

- $|f_n(x)| \leq M_n$ for all $n \geq 1$ and all $x \in A$, and
- $\displaystyle\sum_{n=1}^{\infty} M_n$ converges.

Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely and uniformly on A.

Example 2

Use the Peano-Baker series to compute the solution:

$$\dot{x} = A(t)x, \quad A(t) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}. \tag{10}$$

Answer: We have

$$\begin{aligned}
\Delta_1 &= \int_{t_0}^{t} A(s)ds = \begin{bmatrix} 0 & \frac{t^2-t_0^2}{2} \\ 0 & 0 \end{bmatrix} \\
\Delta_2 &= \int_{t_0}^{t} \begin{bmatrix} 0 & s_1 \\ 0 & 0 \end{bmatrix} \int_{t_0}^{s_1} \begin{bmatrix} 0 & s_2 \\ 0 & 0 \end{bmatrix} ds_2 ds_1 = 0 \\
\Delta_k &= 0, \quad k \geq 2.
\end{aligned}$$

Therefore,

$$x(t) = \begin{bmatrix} 1 & \frac{t^2-t_0^2}{2} \\ 0 & 1 \end{bmatrix} x_0. \tag{11}$$

State Transition Matrix

The solution is given using the Peano-Baker series as

$$x(t) = S_\infty(t)x_0.$$

Definition 7 (State Transition Matrix) We define the Peano-Baker series as a function of two variables as

$$\Phi(t,\tau) = I + \int_\tau^t A(s)ds + \int_\tau^t A(s_1)\int_\tau^{s_1} A(s_2)ds_2 ds_1 + \dots$$

which is as the state transition matrix. Similar to DT, we have

$$x(t) = \Phi(t,\tau)x(\tau). \tag{12}$$

Properties of STM (Continuous-time)

Proposition 1 *State transition matrices in continuous time satisfy the following:*

1. *It is the solution to*

$$\frac{d}{dt}\Phi(t,t_0) = A(t)\Phi(t,t_0), \quad \Phi(t_0,t_0) = I, \tag{13}$$

such that the i*-column of* $\Phi(t,t_0) \in \mathbb{R}^{n\times n}$ *corresponds to a solution of* $\dot{x} = A(t)x$ *with* $x_0 = e_i$*.*

2. *For* t_0, t_1, t_2*, we have*

$$\Phi(t_2,t_0) = \Phi(t_2,t_1)\Phi(t_1,t_0). \tag{14}$$

3. *For any* t, s*,* $\Phi(t,s)$ *is invertible.*

Proof

1. Omitted.
2. Consider $x_0^1, \ldots, x_0^n \in \mathbb{R}^n$ and assume that

$$X_0 = [x_0^1, \ldots, x_0^n]$$

is full rank. We collect the solutions $x^i(t)$ in a matrix $X(t) := [x^1(t), \ldots, x^n(t)]$. Then, $X(t) = \Phi(t, t_0)X_0$, thus

$$\Phi(t, t_0) = X(t)X_0^{-1}. \qquad (15)$$

Now selecting $X_0 = I$, we have $X(t_1) = \Phi(t_1, t_0)$ and $X(t_2) = \Phi(t_2, t_0)$. Therefore, $\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0)$.

3. $\Phi(s, s) = I = \Phi(s, t)\Phi(t, s)$, and thus

$$\Phi(t, s)^{-1} = \Phi(s, t). \qquad (16)$$

■

Proposition 2 *(Determinant) For the state transition matrix* $\Phi(t, t_0)$,

$$\det[\Phi(t, t_0)] = \exp\left(\int_{t_0}^{t} \mathit{Tr}[A(s)]ds\right) \tag{17}$$

Proof.

Define $f(t) = \det[\Phi(t, t_0)]$ and $g(t) = \exp\left(\int_{t_0}^{t} \mathrm{Tr}[A(s)]ds\right)$. We show $f \equiv g$ by verifying that they satisfy the same linear ODE, and then using the uniqueness property.

First, $f(t_0) = g(t_0) = 1$. They also satisfy the same ODE:

$$\frac{d}{dt}\exp\left(\int_{t_0}^{t} \mathrm{Tr}[A(s)]ds\right) = \mathrm{Tr}[A(t)]\exp\left(\int_{t_0}^{t} \mathrm{Tr}[A(s)]ds\right) \tag{18}$$

or equivalently $\dot{g} = \mathrm{Tr}[A(t)]g$.

On the other hand,

$$\begin{aligned}\frac{d}{dt}\det[\Phi(t,t_0)] &= \mathsf{Tr}\left[\mathsf{adj}(\Phi(t,t_0))\frac{d}{dt}\Phi(t,t_0)\right] \\ &= \mathsf{Tr}\left[\mathsf{adj}(\Phi(t,t_0))A(t)\Phi(t,t_0)\right] \\ &= \mathsf{Tr}\left[\Phi(t,t_0)\mathsf{adj}(\Phi(t,t_0))A(t)\right] \\ &= \mathsf{Tr}[A(t)]\det[\Phi(t,t_0)],\end{aligned} \tag{19}$$

that is $\dot{f} = \mathsf{Tr}[A(t)]f$. Comparing the dynamics of f and g and the initial conditions, it completes the proof. ■

Hint: $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ For a scalar t

$$\frac{d}{dt}\det[A] = \mathsf{Tr}\left[\mathsf{adj}(A)\frac{dA}{dt}\right]. \tag{20}$$

adj is the adjoint matrix, satisfying $\mathsf{adj}(A)A = \det(A)\cdot I$. The trace has the operation $\mathsf{Tr}[ABC] = \mathsf{Tr}[CBA] = \mathsf{Tr}[ACB]$.

4. Complete Solutions in Continuous Time

Change of State Variables/Coordinates

Often changes of state variables are of interest. Consider again

$$\dot{x} = A(t)x, \quad x(t_0) = x_0. \tag{21}$$

A *new state vector* is defined by

$$z(t) = P^{-1}(t)x(t)$$

with the $n \times n$ matrix $P(t)$ invertible and continuously differentiable. Calculate the time derivative of $x = P(t)z$, we have

$$\dot{x} = P(t)\dot{z} + \dot{P}(t)z.$$

We obtain the dynamical equation of z:

$$\dot{z} = \underbrace{\left[P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t)\right]}_{:=F(t)} z(t), \quad z(t_0) = P^{-1}(t_0)x_0.$$

Proposition 3 *(Property of Coordinate Change) Suppose $P(t)$ is continuously-differentiable, $n \times n$ matrix function s.t. $P^{-1}(t)$ exists for every value of t. Then, the transition matrix for*

$$F(t) = P^{-1}(t)A(t)P(t) - P^{-1}(t)\dot{P}(t) \tag{22}$$

is given by

$$\Phi_F(t,s) = P^{-1}(t)\Phi_A(t,s)P(s).$$

Proof.

From $x = P(t)z$ and $x(t) = \Phi_A(t,t_0)x_0$, we have

$$P(t)z(t) = \Phi_A(t,s)P(s)z(s) \implies z(t) = \underbrace{P^{-1}(t)\Phi_A(t,s)P(s)}_{=\Phi_F(t,s)} z(s).$$

■

Adjoint Equation

Given an ODE $\dot{x} = f(x, t)$, we call another ODE $\dot{p} = g(p, t)$ its *adjoint equation* provided that for any two solutions $x(t), p(t)$, the inner product $p^\top(t)x(t) \equiv$ const.

Theorem 8 *The adjoint equation of $\dot{x} = A(t)x$, with the STM $\Phi(t, t_0)$, is*

$$\dot{p} = -A^\top(t)p.$$

The STM of the adjoint equation is given by $\Phi^\top(t_0, t)$, i.e.

$$\frac{d}{dt}\Phi^\top(t_0, t) = -A^\top(t)\Phi^\top(t_0, t) \implies \frac{d}{dt}\Phi(t_0, t) = -\Phi(t_0, t)A(t).$$

Proof.

To obtain the adjoint equation, we have

$$\frac{d}{dt}(p^\top x) = \dot{p}^\top x + p^\top \dot{x} = -p^\top A x + p^\top A x = 0.$$

For to STM, according to the definition we have

$$p^\top(t)x(t) = p^\top(t_0)x_0 \implies [p^\top(t)\Phi(t,t_0) - p^\top(t_0)]x_0 = 0.$$

The above holds for all $x_0 \in \mathbb{R}^n$, thus $p^\top(t)\Phi(t,t_0) - p^\top(t_0) = 0$,

$$p(t) = \Phi^{-\top}(t,t_0)p(t_0) = \Phi(t_0,t)^\top p(t_0).$$

with $\Phi^{-\top} := (\Phi^{-1})^\top$. Finally, from

$$\frac{d}{dt}\left\{\Phi^{-1}(t,t_0)\Phi(t,t_0)\right\} = \frac{d}{dt}I = 0,$$

we can verify the dynamics of $\Phi(t_0,t)$. ∎

Self-Adjoint

Definition 9 An ODE $\dot{x} = A(t)x$ is self-adjoint if $A(t) = -A^\top(t)$ (skew-symmetric) for all t. (Important in mechanics!)

Theorem 10 *The state transition matrix Φ of a self-adjoint system is orthogonal, i.e., $\Phi(t,t_0)^\top\Phi(t,t_0) = I$, $\forall t_0, t$.*

Proof.

$$\frac{d}{dt}\left\{\Phi^\top(t,t_0)\Phi(t,t_0)\right\} = \Phi^\top(t,t_0)\underbrace{A^\top(t)}_{-A(t)}\Phi(t,t_0) + \Phi^\top(t,t_0)A(t)\Phi(t,t_0) = 0.$$

Therefore,

$$\Phi^\top(t,t_0)\Phi(t,t_0) = \Phi(t_0,t_0)^\top\Phi(t_0,t_0) = I.$$

∎

Example 3: Rotation Matrix

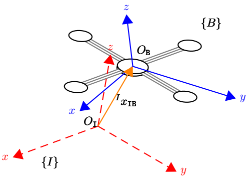
The rotation matrix $R \in \mathsf{SO}(3) \subset \mathbb{R}^{3\times 3}$, which admits the dynamics

$$\dot{R} = A(t)R, \tag{23}$$

with the skew-symmetric matrix

$$A = [\omega]_\times := \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}.$$

The rotation matrix R is given by $R(t) = \Phi(t, t_0)R(t_0)$, and the STM $\Phi(t, t_0)$ is orthogonal.



Nonhomogeneous/Forced Systems

- Similar to the DT case, even though the methods are different.

Consider

$$\dot{x} = A(t)x(t) + g(t), \quad x(t_0) = x_0,$$

in which we write $g(t) := B(t)u(t)$.

- To find a solution, follow the "variation of constants" method for ODEs. Let us change the variable

$$z(t) := \Phi(t, t_0)^{-1}x(t).$$

Its dynamics is

$$\dot{x} = Ax + \Phi(t, t_0)\dot{z}, \quad x(t_0) = z(t_0) = x_0.$$

Then, it yields

$$\Phi(t, t_0)\dot{z} = g(t) \implies \dot{z} = \Phi(t, t_0)^{-1}g(t)$$

Complete Solutions for Nonhomogeneous/Forced Systems

- Integrating it yields

$$z(t) = x_0 + \int_{t_0}^{t} \Phi(t_0, s) g(s) ds.$$

Finally, a solution is

$$x(t) = \underbrace{\Phi(t, t_0) x_0}_{\text{Zero-Input Response}} + \underbrace{\int_{t_0}^{t} \Phi(t, s) B(s) u(s) ds}_{\text{Zero-State Response}}.$$

Note: This solution is again unique!

5. LTI Systems

Matrix Exponential

- If A is constant, the Peano-Baker series give for the STM

$$\begin{aligned}\Phi_A(t,t_0) &= \sum_{k=0}^{\infty} \Delta_k \\ \Delta_k &= \int_{t_0}^{t} A(s_1)\ldots\int_{t_0}^{s_{k-1}} A(s_k)ds_k\ldots ds_1,\ k\geq 1 \\ &= A^k \int_{t_0}^{t}\int_{t_0}^{s_1}\ldots\int_{t_0}^{s_{k-1}} ds_k\ldots ds_1 \\ &= A^k\frac{(t-t_0)^k}{k!} \qquad \text{(via induction)}\end{aligned}$$

- We define the limit of the series as the exponential of a matrix

$$\exp(M) := \sum_{k=0}^{\infty}\frac{M^k}{k!} \qquad \text{(also written as } e^M\text{)}$$

Properties of Matrix Exponential

Proposition 4

1. *For LTI systems, the STM depends on a single variable* $(t - t_0)$*:*

$$\Phi(t, t_0) = \Phi(t - t_0, 0) = \exp(A(t - t_0)).$$

2. *Diagonal matrix:*

$$\exp\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}$$

3. $Ae^{At} = e^{At}A$

The last can be verified directly from series.

Other Properties

- $\exp(At)$ is the unique solution of of $\frac{d}{dt}e^{At} = Ae^{At}$, $e^{A\cdot 0} = I$.
- $\exp(A(s+t)) = \exp(As)\exp(At)$, $\forall s, t \in \mathbb{R}$.
- $\det[\exp(A(t-t_0))] = \exp(\mathsf{Tr}[A](t-t_0)) \implies$ $\det[\exp(A)] = \exp(\mathsf{Tr}[A])$.
- $\exp(At)^{-1} = \exp(-At) \implies \exp(A)^{-1} = \exp(-A)$
- $\exp(P^{-1}AP) = P^{-1}\exp(A)P$, $\forall P$ invertible.
- Solution to the LTI system $\dot{x} = Ax + Bu, \;\; x(t_0) = x_0$ is

$$x(t) = \underbrace{e^{A(t-t_0)}x_0}_{\text{Zero-Input Response}} + \underbrace{\int_{t_0}^{t} e^{A(t-s)}Bu(s)ds}_{\text{Zero-State Response}}.$$

Further Properties

Proposition 5 *For all* t, $\exp((A+B)t) = \exp(At)\exp(Bt)$ *with* $A, B \in \mathbb{R}^{n\times n}$ *if and only if* $AB = BA$.

Proof.

$(\Leftarrow)$ For $t = 0$, LHS $= I =$ RHS. The dynamics is given by

$$\frac{d}{dt}\text{LHS} = (A+B)e^{(A+B)t}$$
$$\frac{d}{dt}\text{RHS} = Ae^{At}e^{Bt} + e^{At}Be^{Bt} = (A+B)e^{(A+B)t}.$$

By the solution uniqueness to $\dot{X} = (A+B)X,\ X(0) = I$, we verify LHS $=$ RHS

($\Rightarrow$) Suppose $e^{(A+B)t} = e^{At}e^{Bt}$, $\forall t$. Then, applying $\frac{d}{dt}[\cdot]$:

$$\begin{aligned}(A+B)e^{(A+B)t} &= Ae^{At}e^{Bt} + e^{At}Be^{Bt}.\\ (A+B)^2e^{(A+B)t} &= A^2e^{At}e^{Bt} + Ae^{At}Be^{Bt} + Ae^{At}Be^{Bt}\\ &\qquad + e^{At}B^2e^{Bt}\end{aligned}$$

It yields

$$A^2 + AB + BA + B^2 = A^2 + AB + AB + B^2.$$

Therefore, we have

$$AB = BA.$$

■

Further Properties

Proposition 6 *There exist* n *analytic scalar functions* $\alpha_0(t), \ldots, \alpha_{n-1}(t)$ *such that*

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k(t) A^k, \ \forall t \in \mathbb{R}. \tag{24}$$

Proof

From Cayley-Hamilton theorem, $A^n = -a_{n-1}A^{n-1} - \ldots - a_1 A - a_0 I$. By immediate induction, for $k \geq n+1$

$$A^k = \tilde{a}_{n-1}(k)A^{n-1} + \ldots \tilde{a}_1(k)A + \tilde{a}_0(k)I.$$

Cayley-Hamilton Theorem. For any square matrix A, we have $A^n + a_1 A^{n-1} + \ldots + a_{n-1}A + a_n I = 0$, i.e., every matrix satisfies its own characteristic equation.

Proof (cont'd)

Then,

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i=0}^{n-1} \tilde{a}_i(k) A^i$$
$$= \sum_{i=0}^{n-1} \left(\underbrace{\sum_{k=0}^{\infty} \frac{t^k \tilde{a}_i(k)}{k!}}_{\alpha_i(t)} \right) A^i. \tag{25}$$

The series α_i is uniformly absolutely convergent. ■

6. Analytic Computation of Matrix Exponential

Method 1: Definition

Use the definition

$$\exp(M) = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

and Cayley-Hamilton theorem.

Method 2: Jordan Normal Form

Particular Case: Diagonalizable Matrices

- If A is diagonalizable, then there exists $P \in \mathbb{C}^{n \times n}$ such that $A = P \Lambda P^{-1}$. We have

$$\exp(A) = P \exp(\Lambda) P^{-1},$$

with

$$\exp\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}$$

- Hence,

$$\boxed{\exp(A) = P \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix} P^{-1}}$$

General Case

More generally, every matrix $A \in \mathbb{C}^{n \times n}$ admits a decomposition $A = PJP^{-1}$, called Jordan Normal Form, with

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_{k} \end{bmatrix}.$$

J_i is the i-th Jordan block associated with eigenvalue λ_i of

$$J_i = \lambda_i \quad \text{or} \quad J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}.$$

Then,

$$\exp(At) = P\exp(Jt)P^{-1}, \qquad \begin{bmatrix} \exp(J_1 t) & & \\ & \ddots & \\ & & \exp(J_k t) \end{bmatrix}.$$

J_i can be decomposed as

$$J_i = \lambda_i I + N_i.$$

We call $N_i \in \mathbb{R}^{d_i \times d_i}$ nilpotent, which commutes with I. Thus,

$$\exp(J_i t) = \exp(\lambda_i t)\exp(N_i t).$$

According to the structure of N_i, we have $N_i^{d_i} = 0$, thus

$$\boxed{\exp(tN_i) = I + tN_i + \ldots + \frac{N_i^{d_i-1}}{(d_i-1)!}t^{d_i-1}.}$$

Method 3: Inverse Laplace Transform

For LTI systems, analysis via Laplace transform becomes available.

Proposition 7 $\exp(At) = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$.

Proof.

$\exp(At)$ is the solution at t of $\dot{X} = AX,\ X(0) = I$. Taking the Laplace transform $\mathcal{L}$ of this ODE:

$$s\hat{X}(s) - I = A\hat{X}(s).$$

Then,

$$(sI - A)\hat{X}(s) = I \implies \hat{X}(s) = (sI - A)^{-1}$$

Therefore, $X(t) = \mathcal{L}^{-1}((sI - A)^{-1})$. By uniqueness of the solution, we have $e^{At} = \mathcal{L}^{-1}((sI - A)^{-1})$. ■

This gives another way of computing $\exp(At)$, via tables of inverse Laplace transforms.

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} \mathsf{adj}(sI - A)$$

The characteristic polynomial

$$\det(sI - A) =: \chi_n(s) = (s - \lambda_1)^{m_1} \dots (s - \lambda_k)^{m_k}.$$

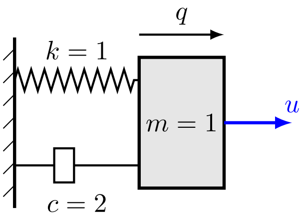
We need to compute the inverse Laplace transform of expressions of the form

$$\frac{A_1}{s - p_j} + \frac{A_2}{(s - p_j)^2} + \dots + \frac{A_{mj}}{(s - p_j)^{mj}}$$

Tip:

$$\mathcal{L}^{-1}\left\{\frac{\alpha}{(s - \lambda)^m}\right\} = \alpha \frac{t^{m-1}}{(m-1)!} e^{\lambda t}.$$

Example 4: Mass-Spring-Damper



$$m\ddot{q}+c\dot{q}+kq = u, \qquad m=1,\ c=2,\ k=1.$$

With $x_1 = q$ and $x_2 = \dot{q}$,

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

Matrix exponential:

$$(sI-A)^{-1} = \frac{1}{(s+1)^2}\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix} = \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ -\frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$$

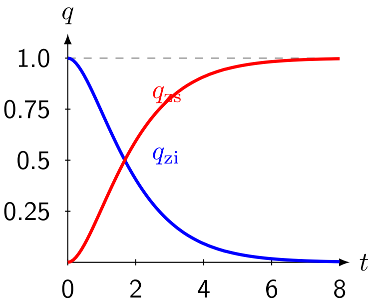
$$e^{At} = \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

Release from $q(0) = 1$, $\dot{q}(0) = 0$, $u = 0$:

$$q_{\mathrm{zi}}(t) = (1+t)e^{-t}.$$

Apply a unit step force from rest:

$$q_{\mathrm{zs}}(t) = \int_0^t \begin{bmatrix} 1 & 0 \end{bmatrix} e^{A(t-s)} B \, ds = 1 - (1+t)e^{-t}.$$



$$q(t) = \underbrace{q_{\mathrm{zi}}(t)}_{\text{stored energy}} + \underbrace{q_{\mathrm{zs}}(t)}_{\text{applied force}}.$$

The same STM governs both responses; the input only determines how copies of that natural response are accumulated.

Example 5: Harmonic Oscillator

Consider the harmonic oscillator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x, \quad x \in \mathbb{R}^2.$$

- A simple calculation yields

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} = \frac{1}{s^2 + 1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}$$

- Partial fraction expansion and the Laplace transform table in Lecture 1 can be used to obtain

$$e^{At} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.$$

7. Full Response

Full Input-Output Response

Consider the LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ y &= Cx + Du. \end{aligned}$$

The state solution is $x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds$, and thus the output solution is

$$y(t) = \underbrace{Ce^{At}x_0}_{\text{zero-input response}} + \underbrace{\int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t)}_{\text{zero-state response}}$$

Transfer Matrices from State-Space Models

Taking Laplace transform

$$\begin{aligned} s\hat{X}(s) - x(0) &= A\hat{X}(s) + B\hat{U}(s) \\ \hat{Y}(s) &= C\hat{X}(s) + D\hat{U}(s). \end{aligned}$$

Then,

$$\begin{aligned} \hat{X}(s) &= (sI - A)^{-1}x_0 + (sI - A)^{-1}B\hat{U}(s) \\ \hat{Y}(s) &= C(sI - A)^{-1}x_0 + \underbrace{\left(C(sI - A)^{-1}B + D\right)}_{\text{Transfer Function/Matrix}} \hat{U}(s) \end{aligned}$$

The transfer function/matrix of the system $\dot{x} = Ax + Bu,\ y = Cx + Du$ is

$$H(s) = C(sI - A)^{-1}B + D$$

satisfying $\hat{Y}(s) = H(s)\hat{U}(s)$.

Some Other Tractable Cases

Proposition 8 *For a time-varying $A(t) \in \mathbb{R}^{n \times n}$, the STM $\Phi(t, t_0)$ has compact forms in the following cases:*

1. *(Integral commutative) If* $A(t) \int_{t_0}^{t} A(s) ds = \int_{t_0}^{t} A(s) ds A(t)$, $\forall t$, *then*

$$\Phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(s) ds\right).$$

(Not true in general!)

2. *If $A(t) = a(t)$ is scalar, then* $\Phi(t, t_0) = e^{\int_{t_0}^{t} a(s) ds}$.
3. *If $A(t)$ is diagonal, i.e.* $A(t) = \text{diag}(a_1(t), \ldots, a_n(t))$, *then*

$$\Phi(t, t_0) = \begin{bmatrix} e^{\int_{t_0}^{t} a_1(s) ds} & & \\ & e^{\int_{t_0}^{t} a_2(s) ds} & \\ & & e^{\int_{t_0}^{t} a_3(s) ds} \end{bmatrix}.$$

Exercise & Reading Materials

Exercise

For a constant $A \in \mathbb{R}^{n\times n}$, solve

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ tA & A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \tag{26}$$

Reading Materials

1. Linear Periodic Systems and Floquet Theory

Linear Periodic Systems ($\star$)

Definition 11 (T-periodic) An $n \times n$ continuous matrix $A(t)$ is called T-periodic if $\exists\, T > 0$ such that

$$A(t+T) = A(t), \quad \forall t.$$

Theorem 12 *(Floquet Decomposition) The STM for a T-periodic matrix $A(t)$ can be decomposed as*

$$\Phi(t,\tau) = P(t)\exp(Rt)P^{-1}(\tau),$$

where R is a constant (possibly complex) $n \times n$ matrix, and $P(t)$ is continuously differentiable, T-periodic, $n \times n$ matrix that is invertible for all t.

Its proof is left for your homework.

Example 6: Floquet Decomposition

- Consider the matrix

$$A(t) = \begin{bmatrix} -1 & 0 \\ -\cos(t) & 0 \end{bmatrix}$$

that is 2π-periodic, with the STM

$$\Phi(t,0) = \begin{bmatrix} e^{-t} & 0 \\ -\frac{1}{2} + e^{-t}\frac{\cos t - \sin t}{2} & 1 \end{bmatrix}$$

- We may verify that

$$R = \frac{1}{T}\ln\Phi(2\pi,0) = \begin{bmatrix} -1 & 0 \\ -\frac{1}{2} & 0 \end{bmatrix},\ P(t) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2}(-1+\cos(t)-\sin(t)) & 1 \end{bmatrix}$$

- Floquet decomposition:

$$\Phi(t,0) = P(t)e^{-Rt}P(0).$$

Sources and Further Reading

- Existence and uniqueness of ODE solutions: We follow standard results in nonlinear systems theory, particularly Sastry.[1]
- State-transition matrices and system responses: The discrete- and continuous-time solution formulas, state-transition matrices, the Peano–Baker series, and variation of constants draw primarily on Rugh and (Antsaklis and Michel).
- Coordinate transformations and adjoint systems: This follows Rugh and (Callier and Desoer).
- LTI systems and matrix exponentials: The matrix exponential, its computation using the Cayley–Hamilton theorem, Jordan form, and inverse Laplace transforms, as well as the complete input–output response and transfer matrix, draw on Rugh, (Callier and Desoer), and (Antsaklis and Michel).
- Examples and further topics: The sloshing-water example is adapted from Welch,[2] while the remaining examples were re-organized by the instructor.

[1] S. Sastry, *Nonlinear Systems: Analysis, Stability, and Control*, Springer, 1999, Ch. 2.

[2] G. Welch, "Dynamic and Measurement Models," UNC–Chapel Hill COMP 145 course notes, 2001, Secs. 1.1.3 and 1.2.3.