ELE6202E - Multivariable Systems

Lecture 4: Internal/Lyapunov Stability

Fall 2026

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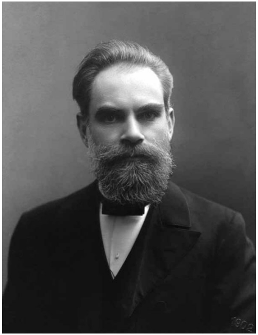
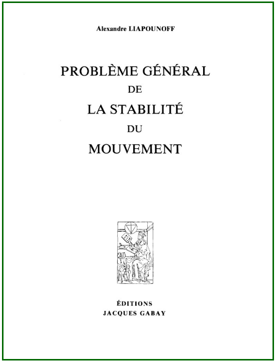
POLYTECHNIQUE MONTRÉAL



Aleksandr Mikhailovich Lyapunov (1857 – 1918)

Russian mathematician, mechanician and physicist, known for developing the stability theory of a dynamical system, as well as for many contributions to mathematical physics and probability theory.

- Master's thesis (proposed by Chebyshev, 1884): On the stability of ellipsoidal forms of rotating fluids.
- Seminal memoir "Obshchaya zadacha ob ustoichivosti dvizheniya" (The general problem of the stability of motion, in Russian)
- Translated to French by Université de Toulouse: Problème General de la Stabilité du Mouvement (1908)



Contents

1. Definitions of Stability (Continuous Time)

Zero Equilibrium

Definition 1 (Equilibrium) For the unforced nonlinear system $\dot{x} = f(x,t)$, $x_\star$ is an equilibrium at $t_\star$ if

$$f(x_\star, t) = 0, \quad \forall t \geq t_\star.$$

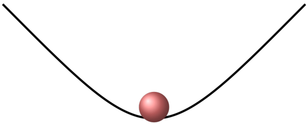
Then, $x(t) \equiv x_\star$ for all $t \geq t_\star$ is a solution.

- By change of variable in $t = t_\star + \tilde{t}$ and $x = x_\star + \tilde{x}$, we have $\dot{\tilde{x}} = f(\tilde{x} + x_\star, \tilde{t} + t_\star)$, which shifts the equilibrium to $\mathbf{0}$ at $\tilde{t} = 0$. Without loss of generality, we consider $t_\star = 0$ and $x_\star = 0$.
- For the linear system $\dot{x} = A(t)x$, the origin is always an equilibrium, and other possible equilibria must be in the set $\cap_{t \geq t_0} N(A(t))$.

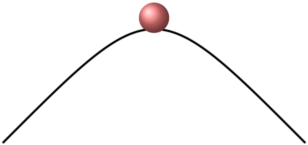
Internal Stability

Internal stability deals with boundedness properties and asymptotic behavior as $t \to \infty$ of solutions of the zero-input (unforced) state-space model:

$$\dot{x} = A(t)x(t), \quad x(t_0) = x_0.$$



(a) Stable Situation



(b) Unstable Situation

Definition 2 (Lyapunov Stability) Denote the solution of $\dot{x} = A(t)x,\ x(t_0) = x_0$ as $X(t; t_0, x_0) = \Phi(t, t_0)x_0$. The origin is

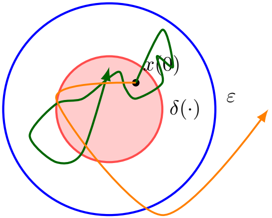
- stable if $\forall \epsilon > 0,\ \forall t_0 \geq 0,\ \forall t \geq t_0,\ \exists \delta(\varepsilon, t_0)$ s.t.

$$\|x_0\| < \delta(\varepsilon, t_0) \implies \|X(t; t_0, x_0)\| < \varepsilon,\ \forall t \geq t_0;$$

- uniformly stable if $\forall \varepsilon > 0,\ \exists \delta(\varepsilon),\ \forall t_0$ s.t.

$$\|x_0\| < \delta(\varepsilon) \implies \|X(t; t_0, x_0)\| < \varepsilon,\ \forall t \geq t_0;$$

- unstable if it is not stable.



Remarks

- This is a general definition of Lyapunov/internal stability, which is also valid for nonlinear systems.
- For LTI systems, uniform stability is equivalent to stability.
- Small perturbation of x_0 about 0 yields small perturbation in trajectories.
- Lyapunov stability for (finite-dimensional) LTV systems can be equivalently defined as

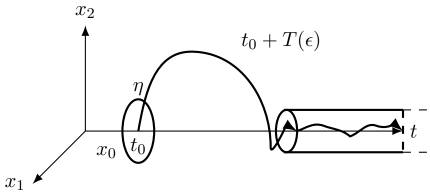
  "The equilibrium 0 is stable if $\forall x_0 \in \mathbb{R}^n$, $\forall t_0 \in \mathbb{R}$, $\forall t \geq t_0$, the map $t \to \Phi(t, t_0)x_0$ is bounded."

Attractivity

Definition 3 (Attractive) The zero equilibrium of the system $\dot{x} = A(t)x,\ x(t_0) = x_0$ is attractive if $\forall t_0 \geq 0,\ \exists \eta(t_0) > 0$, s.t.

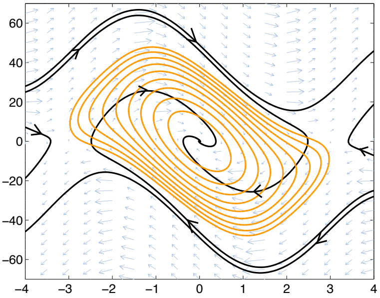
$$\|x_0\| < \eta(t_0) \implies \lim_{t \to \infty} X(t; t_0, x_0) = 0.$$

If η is a constant scalar independent of initial time t_0, we call it uniformly attractive.



Domain of Attraction

Definition 4 The set of all x_0 such that $X(t; t_0, x_0) \to 0$ as $t \to +\infty$ is called the domain of attraction of the equilibrium 0 at t_0.



Asymptotic and Exponential Stability

Definition 5 (Asymptotic and Exponential Stability) The zero equilibrium for

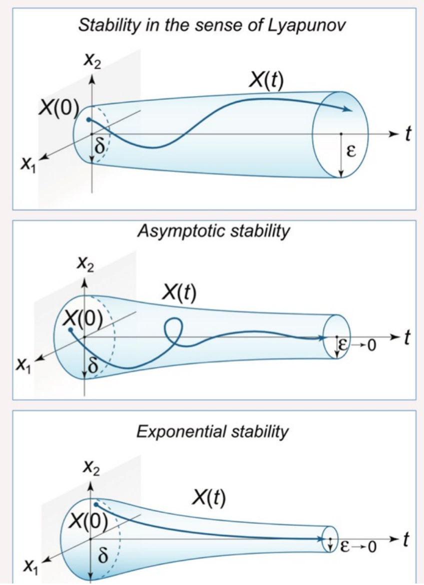
$$\dot{x} = A(t)x,\ x(t_0) = x_0$$

is

- asymptotically stable if it is Lyapunov stable and attractive;
- uniformly asymptotically stable (UAS) if it is uniformly stable and uniformly attractive;
- uniformly exponentially stable if $\exists r, a, b > 0$ s.t.

$$\|X(t_0 + t; t_0, x_0)\| \leq a\|x_0\|e^{-bt},\ \forall t_0, t \geq 0,\ \forall \|x_0\| \leq r$$

Different Concepts of Stability



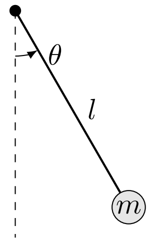
Example 1: Pendulum

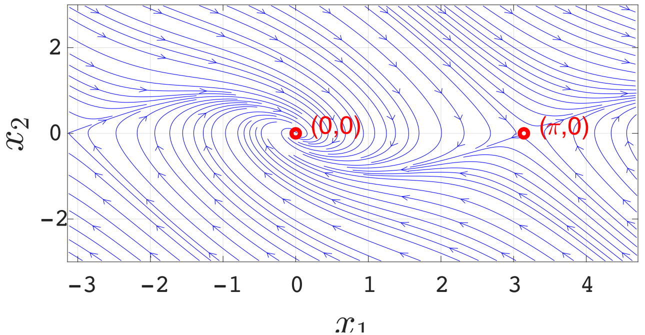
The state-space model is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$$

with the state $x = \text{col}(\theta, \dot{\theta})$





Example 2

Consider the LTV system

$$\dot{x} = a(t)x$$

whose STM is $\Phi(t, t_0) = \exp(\int_{t_0}^{t} a(s)ds$, for the following cases:

- $a(t) = -1$: UES;
- $at(t) = 0$: uniformly stable but not attractive;
- $a(t) = -1/(1+t)$: asymptotically stable but not uniformly stable;
- $a(t) = 1/(1+t)$: unstable.

Remarks

- Exponential stability $\implies$ UAS.
- For nonlinear systems, we can have attractive equilibria that are **not** stable.
- For time invariant systems and periodic systems, UAS $\iff$ A.S.
- The above definitions are local (i.e. depending on initial conditions). We will be interested in what happens for any initial conditions (global).

Global Stability

Definition 6 (Global Stability) The zero equilibrium is

- globally asymptotically stable (GAS) if it is stable and $\forall t_0, x_0$, $\lim_{t\to\infty} x(t; t_0, x_0) = 0$;
- uniformly globally asymptotically stable (UGAS) if (1) it is uniformly stable; and (2) $\forall \beta$ (arbitrarily large) and $\forall \varepsilon$ (arbitrarily small), $\exists T = T(\beta, \varepsilon)$ s.t.

$$\|x_0\| < \beta,\ t_0 \geq 0 \quad \Longrightarrow \quad \|X(t_0 + t; t_0, x_0)\| < \varepsilon,\ \forall t \geq T.$$

- globally exponentially stable (GES), if $\exists a, b > 0$ s.t.

$$\|X(t_0 + t; t_0, x_0)\| \leq a\|x_0\|e^{-bt},\ \forall t_0, t \geq 0,\ \forall x_0 \in \mathbb{R}^n.$$

Note: If an equilibrium is GAS, there is no other equilibrium.

Example. (Antsaklis p. 451)

The system

$$\dot{x} = -\frac{1}{t+1}x, \quad x(t_0) = x_0 \in \mathbb{R}$$

with $t_0 \geq 0$ has a unique solution

$$X(t; t_0, x_0) = \frac{t_0 + 1}{t + 1}x_0 \quad \Longrightarrow \quad \Phi(t, t_0) = \frac{t_0 + 1}{t + 1}$$

$\left(\frac{\dot{x}}{x} = -\frac{1}{t+1} \implies \ln x - \ln x_0 = \ln(t_0 + 1) - \ln(t + 1)\right)$

The zero equilibrium is uniformly stable. (We will learn how to verify this.) However, it is not uniformly attractive, since

$$\frac{t_0 + 1}{t_0 + T + 1} < \varepsilon \iff T > \frac{1}{\varepsilon}(t_0 + 1)(1 - \varepsilon) \text{ [depends on } t_0]$$

2. Stability Criteria via State Transition Matrix

State Transition Matrix Conditions

Theorem 7 The zero equilibrium for $\dot{x} = A(t)x,\ x(t_0) = x_0$ is

1. uniformly stable (U.S.)
   $\iff \exists\alpha > 0 : \|\Phi(t,\tau)\| \le \alpha, \forall t \ge \tau$;
2. uniformly exponentially stable (UES) $\iff \exists\gamma \ge 1, \lambda > 0$ s.t. $\forall t \ge \tau$
   $$\|\Phi(t,\tau)\| \le \gamma e^{-\lambda(t-\tau)};$$
3. UES $\iff$ UAS [Only true for linear systems.]

Additionally, if $\exists$ finite $\beta > 0$ s.t. $\|A(t)\| < \beta$ for all t. Then, the zero equilbrium is

4. UES $\iff \exists\alpha > 0$ s.t.
   $$\int_\tau^t \|\Phi(t,s)\| ds < \alpha, \quad \forall t \ge \tau.$$

Proof

(1) Uniform stability

($\Leftarrow$) $\|x(t)\| = \|\Phi(t, t_0)x_0\| \leq \|\Phi(t, t_0)\| \|x_0\| \leq \alpha \|x_0\| \implies$ U.S.

($\Rightarrow$) According to the definition $\|M\| = \max_{|r|=1} \|Mr\|$, $\exists r$ such that

$$\|r\| = 1 \quad \text{and} \quad \|M\| = \|Mr\|. \tag{1}$$

Taking $M = \Phi(t_a, t_0)$ and selecting this particular r, this implies

$$\begin{aligned} \|X(t_a; t_0, r)\| &= \|\Phi(t_a, t_0)r\| \\ &= \|\Phi(t_a, t_0)\| \\ &\leq \varepsilon \end{aligned}$$

Since t_0, t_a, r is arbitrary, the proof is completed.

(2), (3), and (4) left for exercise. See (Rugh, p.102). ■

Relations Among Stability Definitions

- For LTV systems,

$$\begin{aligned} \text{UAS} \quad &\Longleftrightarrow \quad \text{UGAS} \\ &\Longleftrightarrow \quad \text{UGES} \\ &\Longleftrightarrow \quad \text{GES} \\ &\Longleftrightarrow \quad \text{Exponential Stability} \\ &\not\Longleftrightarrow \quad \text{Asymptotic Stability} \end{aligned}$$

- For LTI systems, exponential stability is equivalent to the above.
- For nonlinear systems, all these are different.

3. Stability for LTI Systems

Stability for LTI Systems

For LTI system $\dot{x} = Ax,\ x(t_0) = x_0$, the STM is

$$\Phi(t, t_0) = e^{A(t-t_0)}.$$

As noted earlier, the initial instant t_0 is not important anymore, and we take it as $t_0 = 0$. The concept of uniformity vis-à-vis the initial time becomes superfluous for LTI systems.

Corollary 8 The system $\dot{x} = Ax$, $x(0) = x_0$ with constant $A \in \mathbb{R}^{n\times n}$ is GAS if and only if either of the conditions is true:

- $\|e^{At}\| \leq \gamma e^{-\lambda t}$;
- $\int_0^\infty \|e^{As}\| ds < +\infty$.

It would be interesting to determine stability from A directly.

Exponential Stability Criterion for LTI Systems

Theorem 9 *(Stability of LTI Systems) The LTI system*

$$\dot{x} = Ax$$

with $x \in \mathbb{R}^n$ is exponentially stable if and only if A is Hurwitz, i.e.

$$Re[\lambda_i(A)] < 0, \quad i = 1, \ldots, n.$$

Proof

$\Leftarrow$) It is easy to show that

$$e^{At} = \sum_{k=1}^{n} \pi_k(t) e^{\lambda_k t},$$

where $\lambda_k \in \mathbb{C}$ are the eigenvalues of A, and $\pi_k(t)$ are matrix polynomials in t. Hence, if we take matrix norms:

$$\|e^{At}\| \leq \sum_{k=1}^{n} \|\pi_k(t)\| e^{\mathsf{Re}(\lambda_k)t} \leq \sum_{k=1}^{n} \|\pi_k(t)\| e^{-\mu t} = p(t) e^{-\mu t},$$

where $\mu = -\mathsf{max}_k(\mathrm{Re}\,\lambda_k) > 0$, and $p(t) = \sum_{k=1}^{n} \|\pi_k(t)\|$.

Now, since a polynomial grows slower than any growing exponential, we have that for any $\epsilon \in (0, \mu)$, there exists $m(\epsilon) > 0$ s.t.

$$0 \leq p(t) \leq m(\epsilon) e^{\epsilon t}, \quad \forall t \geq 0.$$

Hence,

$$\|e^{At}\| \leq p(t)e^{-\mu t} \leq m(\epsilon)e^{-(\mu-\epsilon)t}.$$

Since $x(t) = e^{At}x(0)$, we have

$$\|x(t)\| \leq \|e^{At}\|\|x(0)\| \leq m(\epsilon)e^{-(\mu-\epsilon)t}\|x(0)\| \to 0, \text{ as } t \to \infty,$$

and so $\dot{x} = Ax$ is exponentially stable.

$\Rightarrow$) We will show this by contrapositive. If one of the eigenvalues of A is not in the open left half plane, then $e^{At} = \sum_{k=1}^{n} \pi_k(t)e^{\lambda_k t}$ does not tend to 0 as $t \to \infty$. Thus $\dot{x} = Ax$ is not exponentially stable. ■

Uniform Stability Criterion

Theorem 10 *(Uniform Stability) For the system $\dot{x} = Ax$ is (uniformly) stable if and only if*

1. *all eigenvalues of A have non-positive real part, and*
2. *every eigenvalue with zero real part has an associated Jordan block of size 1 (i.e., no $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, etc).*

Example. The system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x, \quad x(0) = x_0 =: \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix}, \tag{2}$$

has the unbounded solution $x(t) = \begin{bmatrix} x_{0,1} + x_{0,2}t \\ x_{0,2} \end{bmatrix}$.

Note: For LTI systems, we have only three possible behaviors:

- exponentially stable,
- stable but not asymptotically stable
- unstable.

Exercise

Consider the following system matrices:

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ A_4 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Solution:

1. exponentially stable;
2. stable but not asymptotically stable;
3. unstable (imaginary-axis Jordan block);
4. exponentially stable (despite a nontrivial Jordan block).

4. Lyapunov's Direct Method

Motivation

- Stability conditions on $\Phi_A(s,t)$ for LTV systems are not practical – the difficulty in computing Φ_A.
- Lyapunov's direct method provides sufficient conditions for stability without computing solutions of ODEs.
- It can be checked directly on $A(t)$ by finding "energy-like" functions that decrease along trajectories.
- Lyapunov theory is fundamental in stability theory, especially for nonlinear systems.

In the linear case, we consider the simple version by restricting to quadratic Lyapunov functions.

Quadratic Lyapunov Function

- A quadratic Lyapunov function $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_+$ has the form

$$V(x,t) = x^\top P(t) x,$$

with $P(t) = P(t)^\top \succ 0 \; \forall t$, and assume $P(t) \in C^1$.

> For LTI systems, we additionally assume P is constant.

- For the system $\dot{x} = A(t)x$, consider the Lie derivative of V:

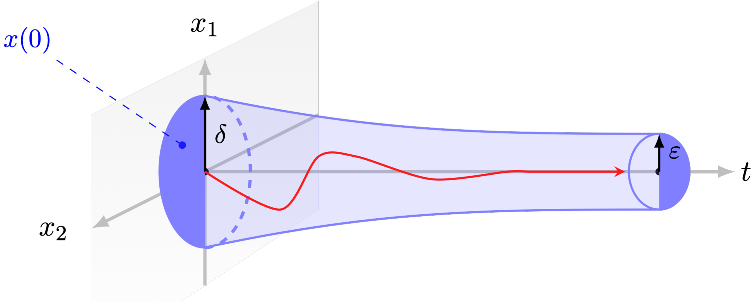
$$\begin{aligned}
\dot{V}(x(t),t) &:= \frac{d}{dt}\left(V(x),t)\right) \\
&= \frac{\partial V}{\partial x}(x,t)\dot{x}(t) + \frac{\partial V}{\partial t}(x,t) \\
&= \dot{x}^\top P x + x^\top P \dot{x} + x^\top \dot{P} x \\
&= x^\top \underbrace{\left[A^\top(t)P(t) + P(t)A + \dot{P}(t)\right]}_{:=Q(t)} x(t)
\end{aligned}$$

Intuition: Energy-like Function

- If $P(t) \succ 0$, the level sets

$$\mathcal{E}_{P(t),c} := \{x \in \mathbb{R}^n : V(x,t) \leq c\}$$

  are ellipsoids centered at the origin.
- If $\dot{V}(x(t),t) \leq 0$, $x(t)$ will stay within the ellipsoid $\mathcal{E}_{P(t),c}$ with $c_0 := V(x_0, t_0)$, thus implying stability.
- If $\dot{V}(x(t),t) < 0$, $x(t)$ will stay in increasingly smaller ellipsoids and converge to 0, implying asymptotic stability.



Theorem 11 Consider the system $\dot{x} = A(t)x, \; x(t_0) = x_0$. If $\exists \eta, \rho > 0$ s.t.

$$\eta I \preceq P(t) \preceq \rho I$$
$$A^\top(t)P(t) + P(t)A(t) + \dot{P}(t) \preceq 0,$$

Then, the given system is uniformly stable. If the second matrix inequality is replaced by

$$A^\top(t)P(t) + P(t)A(t) + \dot{P}(t) + \alpha I \preceq 0, \quad \alpha > 0,$$

then, the system is uniformly asymptotically stable with

$$\|x(t)\| \leq \left(\frac{\rho}{\eta}\right)^{\frac{1}{2}} \exp\left(-\frac{\alpha}{2\rho}(t - t_0)\right) \|x_0\|, \quad \forall t \geq t_0.$$

Note: We usually select $\eta := \inf_t \lambda_{\mathtt{min}}(P(t))$ and $\rho := \sup_t \lambda_{\mathtt{max}}(P(t))$.

Proof

Integrate $\dot{V}$ over time:

$$V(x(t),t) - V(x_0,t_0) = \int_{t_0}^{t} x^\top(s)Q(s)x(s)ds \le 0,$$

with $Q := A^\top P + PA + \dot{P}$. For the U.S. case, we have

$$\eta\|x(t)\|^2 \le V(x(t),t) \le V(x_0,t_0) \le \rho\|x_0\|^2 \implies \|x(t)\| \le \sqrt{\frac{\rho}{\eta}}\|x_0\|,$$

implying uniform stability.

For the case of UAS, we have

$$\frac{d}{dt}\left(x^\top P(t)x\right) \le -\alpha\|x(t)\|^2, \quad \forall t \ge t_0.$$

The first inequality $\eta I \preceq P(t) \preceq \rho I$ means $V(x,t) = x^\top(t)P(t)x(t) \leq \rho\|x(t)\|^2$, and thus

$$-\|x(t)\|^2 \leq -\frac{1}{\rho}x^\top(t)P(t)x(t).$$

It then yields

$$\frac{d}{dt}\left(x^\top(t)P(t)x(t)\right) \leq -\frac{\alpha}{\rho}x^\top(t)Px(t).$$

Using Gronwall inequality (i.e. $\dot{f} \leq \alpha f(t) \implies f(t) \leq f(t_0)e^{\alpha(t-t_0)}$), we have

$$x^\top(t)P(t)x(t) \leq x_0 P(t_0)x_0 e^{-\frac{\alpha}{\rho}(t-t_0)}$$

$$\eta\|x(t)\|^2 \leq x^\top(t)P(t)x(t) \leq x_0^\top P(t_0)x_0 e^{-\frac{\alpha}{\rho}(t-t_0)} \leq \rho\|x_0\|^2 e^{-\frac{\alpha}{\rho}(t-t_0)}$$

It follows that $\|x(t)\| \leq \left(\frac{\rho}{\eta}\right)^{\frac{1}{2}} \exp\left(-\frac{\alpha}{2\rho}(t-t_0)\right)\|x_0\|$. ∎

Example 3

Consider the LTV system $\dot{x} = A(t)x$ with

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 & -a(t) \end{bmatrix},\ a(t) \geq 0.$$

Selecting $P(t) = I$, we have $\eta = \rho = 1$, and

$$A^\top I + IA + 0 = \begin{bmatrix} 0 & 0 \\ 0 & -2a(t) \end{bmatrix} \preceq 0 \iff a(t) \geq 0, \forall t > t_0.$$

Therefore, the given system is uniformly stable.

Remarks

- The above selection does not allow us to conclude UAS.
- It is possible to use another choice of $P(t)$ to get less conservatism. [Sufficient condition depending on the selection of $P(t)$.]
- We can NOT use $\lambda(A(t))$ to determine stability for time-varying systems.
- However, this system may not be UAS. For example, consider the system parameter $a(t) = 2 + e^t > 2 > 0$. A particular solution

$$x(t) = \begin{bmatrix} 1 + e^{-t} \\ -e^{-t} \end{bmatrix}$$

which has

$$\lim_{t \to \infty} x(t) \neq 0_2 \implies \text{Not UAS.}$$

Converse Lyapunov Theorem (LTV)

Theorem 12 *Suppose that the system $\dot{x} = A(t)x$ is UES, and $\exists$ a finite constant α such that $\|A(t)\| \leq \alpha$ for all t. Then,*

$$P(t) = \int_t^\infty \Phi^\top(s,t)\Phi(s,t)ds$$

satisfies all the hypotheses in the Lyapunov theorem for UES.

The proof is given in [Rugh, pp. 120].

Sketch of Proof: Show

1. the well-posedness of $P(t)$;
2. the positiveness of $P(t)$, i.e. $P(t) \succeq \gamma I$;
3. It is a solution to the differential matrix inequality.

Lyapunov Equation

For the LTI system $\dot{x} = Ax$, we consider a constant $P \in \mathbb{R}^{n\times n}_{\succ 0}$ with the time-invariant Lyapunov function

$$V(x) = x^\top P x.$$

The Lyapunov condition reduces into

$$\exists\; P \succ 0 : \qquad A^\top P + PA \prec 0. \tag{3}$$

Equivalently, given a matrix $M \succ 0$, solve the algebraic, linear matrix equation (called Lyapunov equation):

$$A^\top P + PA + M = 0.$$

Lyapunov Theorem for LTI Systems

Theorem 13 Given an $A \in \mathbb{R}^{n \times n}$,

$M, P \succ 0$ satisfy Lyapunov Eq. $\implies \mathrm{Re}\{\lambda_i(A)\} < 0,\ i = 1, \ldots, n.$

Conversely, if all $\lambda_i(A)$ have negative real parts, then $\forall M \in \mathbb{R}^{n \times n}$ symmetric, there exists a unique solution to the Lyapunov equation that is given by

$$P = \int_0^{\infty} e^{A^\top t} M e^{At} dt.$$

Additionally, $M \succ 0 \implies P \succ 0$.

Proof

($\Rightarrow$) The matrices $M, P \succ 0$ satisfy the Lyapunov equation implies the given LTI system is UES. Invoking the fact that UES is equivalent to that all $\lambda_i(A)$ live on the open left plane, we verify the first part.

cont'd

($\Leftarrow$) If $\mathrm{Re}(\lambda_i(A)) < 0$, the integral definition of P is convergent, thus P is well defined.

To show that P is a solution of the Lyapunov equation, we calculate

$$\begin{aligned} A^\top P + PA &= \int_0^\infty A^\top e^{A^\top t} M e^{At} dt + \int_0^\infty e^{A^\top t} M e^{At} A dt \\ &= \int_0^\infty \frac{d}{dt} \left[e^{A^\top t} M e^{At} \right] dt \\ &= e^{A^\top t} M e^{At} \Big|_0^\infty = -M. \end{aligned}$$

To prove this solution is unique, suppose P_a also is a solution. Then

$$(P_a - P) A + A^\top (P_a - P) = 0.$$

But this implies

cont'd.

$$e^{A^\top t}\left(P_a - P\right)Ae^{At} + e^{A^\top t}A^\top\left(P_a - P\right)e^{At} = 0, \quad t \geq 0$$

from which

$$\frac{d}{dt}\left[e^{A^\top t}\left(P_a - P\right)e^{At}\right] = 0, \quad t \geq 0.$$

Integrating both sides from 0 to ∞ gives

$$0 = e^{A^\top t}(P_a - P)e^{At}\Big|_0^\infty = -(P_a - P).$$

That is, $P_a = P$.

Now suppose that $M \succ 0$. Clearly P is symmetric. For a nonzero $x \in \mathbb{R}^n$,

$$x^\top Px = \int_0^\infty x^\top e^{A^\top t}Me^{At}xdt > 0$$

since the integral is a positive scalar function. ■

Solution to Lyapunov Equations

Theorem 14 *A matrix A is Hurwitz if and only if any choice of $M \succ 0$ the Lyapunov equation*

$$A^\top P + PA + M = 0$$

has a unique solution $P \succ 0$.

5. Additional Stability Criteria

Eigenvalue Conditions for LTV Systems ?

Similar to LTI systems, it might be thought that pointwise-in-time eigenvalues of $A(t)$ could be used to characterize Lyapunov stability of the LTV system

$$\dot{x} = A(t)x, \quad x(t_0) = x_0,$$

but this is NOT generally true.

Example (Non-UAS systems with $\mathrm{Re}(\lambda_i(A(t)) < 0$)

$$A(t) = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \quad \Longrightarrow \lambda_1(A) = \lambda_2(A) = -1.$$

Its solution is $x_1(t) = e^{-t}x_1(0) + \frac{1}{2}x_2(0)(e^t - e^{-t}),\ x_2(t) = e^{-t}x_2(0)$. Clearly, x_1 has an unbounded term $e^t x_2(0)$.

Example (Non-UAS with $\mathrm{Re}(\lambda_i(A(t))) < 0$, $\|A(t)\| < \infty$)

$$A(t) = \begin{bmatrix} -1 + \alpha \cos^2 t & 1 - \alpha \sin t \cos t \\ -1 - \alpha \sin t \cos t & -1 + \alpha \sin^2 t \end{bmatrix}$$

where $\alpha > 0$ is constant, and the pointwise eigenvalues are constants, given by

$$\lambda(t) = \lambda = \frac{\alpha - 2 \pm \sqrt{\alpha^2 - 4}}{2}$$

It is not difficult to verify that

$$\Phi(t, 0) = \begin{bmatrix} e^{(\alpha-1)t} \cos t & e^{-t} \sin t \\ -e^{(\alpha-1)t} \sin t & e^{-t} \cos t \end{bmatrix}$$

Thus while the pointwise eigenvalues of $A(t)$ have negative real parts if $0 < \alpha < 2$, the state equation has unbounded solutions if $\alpha > 1$.

Bounds Estimation in LTV Systems

Theorem 15 *(Bounds Estimation) For the LTV system*

$$\dot{x} = A(t)x, \quad x(t_0) = x_0$$

denote the largest and smallest pointwise eigenvalues of $A(t) + A^\top(t)$ *as* λ_m *and* λ_M. *Then, for any* (x_0, t_0) *the solution satisfies for* $t \geq t_0$

$$\|x_0\| \exp\left(\frac{1}{2}\int_{t_0}^{t} \lambda_m(s)ds\right) \leq \|x(t)\| \leq \|x_0\| \exp\left(\frac{1}{2}\int_{t_0}^{t} \lambda_M(s)ds\right).$$

Proof

From the continuity of $A(t)$, we know the well-posedness of λ_m, λ_M. Suppose $x(t)$ is a solution from the initial condition $x(t_0) = x_0$. Using

$$\frac{d}{dt}\|x(t)\|^2 = \frac{d}{dt}\left[x^\top(t)x(t)\right] = x^T(t)\left[A^\top(t) + A(t)\right]x(t)$$

the Rayleigh-Ritz inequality gives

$$\|x\|^2\lambda_m \leq \frac{d}{dt}\|x\|^2 \leq \|x\|^2\lambda_M.$$

Dividing through by $\|x(t)\|^2$ and integrating from t_0 to $t \geq t_0$ yields

$$\int_{t_0}^t \lambda_m(s)ds \leq \ln\|x(t)\|^2 - \ln\|x_0\|^2 \leq \int_{t_0}^t \lambda_M(s)ds.$$

■

Slowly-Varying Systems

Theorem 16 *(Eigenvalue Condition for Slowly-Varying Systems)*
Consider the linear system

$$\dot{x} = A(t)x, \; x(t_0) = x_0$$

with $A(t) \in C^1$. Assume that for all t, $\exists \alpha, \mu > 0$ s.t.

1. $\|A(t)\| < \alpha$;
2. $\mathrm{Re}(\lambda_i(A(t))) \leq -\mu$.

Then, there exists a $\beta > 0$ such that

$$\|\dot{A}(t)\| < \beta \implies \text{UAS.}$$

Its proof is given in [Rugh, pp. 135].

Stability for Perturbed Systems

- Two systems are "close", and one has the stability properties.
- Additive perturbation to the system matrix $A(t)$.

Theorem 17 Suppose the LTV system $\dot{x} = A(t)x$ is uniformly stable. Then, the perturbed system

$$\dot{z} = [A(t) + \Delta(t)]z \tag{4}$$

is uniformly stable if $\exists \beta > 0$ s.t. $\forall \tau$

$$\int_{\tau}^{\infty} \|\Delta(s)\| ds \leq \beta.$$

If the system $\dot{x} = A(t)x$ is UAS and $\|A(t)\| \leq \alpha$, $\forall t$. Then,

$$\exists \beta : \ \|\Delta(t)\| \leq \beta, \ \forall t \quad \Longrightarrow \quad (4) \text{ is UAS.} \tag{5}$$

Proof

For any (z_0, t_0), the solution of the perturbed system satisfies

$$z(t) = \Phi_A(t, t_0)z_0 + \int_{t_0}^{t} \Phi_A(t, s)\Delta(s)z(s)ds$$

where Φ_A denotes the STM of $A(t)$. By the U.S. of $\dot{x} = A(t)x$, $\exists\gamma$ s.t. $\|\Phi_A(t, \tau)\| \leq \gamma$ for all $t \geq \tau$. Therefore, taking norms,

$$\|z(t)\| \leq \gamma\|z_0\| + \int_{t_0}^{t} \gamma\|\Delta(s)\|\|z(s)\|ds,\ t \geq t_0.$$

Applying the Gronwall-Bellman inequality gives

$$\|z(t)\| \leq \gamma\|z_0\|e^{\int_{t_0}^{t} \gamma\|\Delta(s)\|ds}, \quad t \geq t_0.$$

Invoking the bound $\int_{\tau}^{\infty} \|\Delta(s)\|ds \leq \beta$, we have $\|z(t)\| \leq \gamma e^{\gamma\beta}\|z_0\|$ for $t \geq t_0$ and the U.S. of the perturbed system is established. [The UAS result is left for exercise.] ■

UAS with Vanishing Perturbation

Theorem 18 *(Vanishing Perturbation)* If $A(t)$ is UAS, then the uniform asymptotic stability is preserved for any disturbance $\Delta(t)$ satisfies *one* of the following conditions:

- $\lim_{t\to\infty} \Delta(t) = 0_{n\times n}$;
- $\int_{t_0}^{\infty} \|\Delta(s)\| ds < \infty$.

Matrix Measure ($\star$)

Definition 19 Let $\|\cdot\|$ be an induced matrix norm on $\mathbb{C}^{n\times n}$. Then, the corresponding matrix measure (a.k.a. logarithmic norm or Lozinskii measure) is the function $\mu : \mathbb{C}^{n\times n} \to \mathbb{R}$ defined by

$$\mu(A) := \lim_{\varepsilon\to 0^+} \frac{\|I+\varepsilon A\|_i - 1}{\varepsilon}.$$

Matrix measure $\mu(A)$, induced from $\|\cdot\|$, satisfies the properties:

- $\mu(A+B) \leq \mu(A) + \mu(B)$;
- $\mu(A) \leq \|A\|$;
- $-\|A\| \leq -\mu(-A) \leq \mu(A) \leq \|A\|$;
- For $\|\cdot\|_2$, we have

$$\mu(A) = \max_i \left\{ \lambda_i \left(\frac{A+A^*}{2} \right) \right\}.$$

Stability Criteria via Matrix Measure ($\star$)

Theorem 20 The solution of $\dot{x} = A(t)x, \ x(t_0) = x_0$ satisfies

$$\|x_0\| \exp\left(-\int_{t_0}^{t} \mu[-A(s)]ds\right) \leq \|x(t)\| \leq \|x_0\| \exp\left(\int_{t_0}^{t} \mu[A(s)]ds\right).$$

We also have for all $t_0 \geq 0$

- $\exists m(t_0)$: $\int_{t_0}^{t} \mu[A(s)]ds \leq m(t_0), \forall t \geq t_0 \implies$ stability;
- $\exists m_0$: $\int_{t_0}^{t} \mu[A(s)]ds \leq m_0, \forall t \geq t_0 \implies$ uniform stability;
- $\lim_{t\to\infty} \int_{t_0}^{t} \mu[A(s)]ds = -\infty \implies$ asymptotic stability;
- $\exists a: \ \mu[A(t)] \leq -a < 0 \implies$ UAS;
- $\lim_{t\to\infty} \int_{t_0}^{t} \mu[-A(s)]ds = -\infty \implies$ instability.

The proof can be found in [Vidyasagar, pp. 204]. Its extension to nonlinear systems is known as contraction analysis.

Stability Under Coordinate Change ($\star$)

Note that the scalar system

$$\dot{x} = x$$

is unstable, but the change of coordinate $z = e^{-2t}x$ gives the stable equation

$$\dot{z} = -z.$$

This motivates some care when allowing for time varying coordinate changes.

Lyapunov Transformation

An $n \times n$ continuously differentiable matrix function $T(t)$ is called a Lyapunov transformation if there exists $\rho > 0$ s.t.

$$\|T(t)\| \leq \rho, \quad \left\|T(t)^{-1}\right\| \leq \rho \quad \forall t$$

For such a transformation we have

$$\begin{aligned} \left\|\Phi_x\left(t, t_0\right)\right\| &= \left\|T(t) \Phi_z\left(t, t_0\right) T\left(t_0\right)^{-1}\right\| \leq \rho^2\left\|\Phi_z\left(t, t_0\right)\right\| \\ \left\|\Phi_z\left(t, t_0\right)\right\| &= \left\|T(t)^{-1} \Phi_x\left(t, t_0\right) T\left(t_0\right)\right\| \leq \rho^2\left\|\Phi_x\left(t, t_0\right)\right\| \end{aligned}$$

Hence both uniform stability and uniform exponential stability are preserved under a coordinate transformation $x(t)=T(t) z(t)$ defined by a Lyapunov transformation.

6. Reading 1: Lyapunov's Indirect Method ($\star$)

Reading Materials

1. Lyapunov's first/indirect method:
   Given a nonlinear system

$$\dot{x} = f(x,t) \quad \underset{\rightarrow}{\text{linearization}} \quad \delta x = \underbrace{\frac{\partial f}{\partial x}(x_\star, t)}_{:=A(t)} \delta x$$

   to study local stability in a small neighbourhood of an equilibrium point $x_\star$.

2. Discrete-time case (Definitions, Lyapunov theorems, eigenvalue conditions)
3. External stability (BIBO stability)

Lyapunov's Indirect Method

Theorem 21 *(Lyapunov's indirect method) Let* $x = 0$ *be an equilibrium point for*

$$\dot{x} = f(x),$$

where $f : D \to \mathbb{R}^n$ *is continuously differentiable and* D *is a neighborhood of the origin. Let* $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$*. Then,*

1. *The origin is asymptotically stable if* $\mathrm{Re}(\lambda_i) < 0$ *for all eigenvalues of* A*.*
2. *The origin is unstable if* $\mathrm{Re}(\lambda_i) > 0$ *for one or more eigenvalues of* A*.*

Proof

Suppose that A is Hurwitz. Then, for any $Q = Q^\top > 0$, there exists a unique matrix $P = P^\top > 0$ satisfying $A^\top P + PA = -Q$. Hence, $V(x) = x^\top Px$ is a Lyapunov function for the linearized system.

Let us use V as a candidate Lyapunov function for the nonlinear system. Write

$$\dot{x} = f(x) = Ax + \big(f(x) - Ax\big) := Ax + g(x).$$

The derivative of V along the trajectories of the nonlinear system is

$$\begin{aligned}\dot{V} &= \dot{x}^\top Px + x^\top P\dot{x} = \big(Ax + g(x)\big)^\top Px + x^\top P\big(Ax + g(x)\big) \\ &= -x^\top Qx + 2x^\top Pg(x).\end{aligned}$$

Since $g(x) = f(x) - Ax$ and $A = \frac{\partial f}{\partial x}\Big|_{x=0}$, we have

$$\frac{\|g(x)\|_2}{\|x\|_2} \longrightarrow 0 \qquad \text{as } \|x\|_2 \longrightarrow 0.$$

Therefore, for any $\gamma > 0$, there exists $r > 0$ such that $\|g(x)\|_2 < \gamma\|x\|_2$, $\forall\, \|x\|_2 < r$.

Using the Cauchy–Schwarz inequality and the induced matrix norm yields

$$\begin{aligned}\dot V(x) &= -x^\top Qx + 2x^\top Pg(x)\\ &\le -x^\top Qx + 2\|x\|_2\|P\|_2\|g(x)\|_2\\ &< -x^\top Qx + 2\gamma\|P\|_2\|x\|_2^2, \qquad \forall\, \|x\|_2 < r.\end{aligned}$$

Moreover, $x^\top Qx \ge \lambda_{\mathtt{min}}(Q)\|x\|_2^2$. Consequently,

$$\dot V < -\Big(\lambda_{\mathtt{min}}(Q) - 2\gamma\|P\|_2\Big)\|x\|_2^2, \qquad \forall \|x\|_2 < r.$$

Choosing $0 < \gamma < \frac{\lambda_{\mathtt{min}}(Q)}{2\|P\|_2}$ gives

$$\lambda_{\mathtt{min}}(Q) - 2\gamma\|P\|_2 > 0.$$

Thus, $\dot V(x)$ is locally negative definite. Since $V(x) = x^\top Px$ is positive definite, the origin of the nonlinear system is locally asymptotically stable. This proves (1). The proof of (2) follows from the instability theorem. ■

7. Reading 2: Internal Stability for Discrete-Time Systems ($\star$)

Discrete-Time Internal Stability

Consider the zero-input discrete-time linear system

$$x(k+1) = A(k)x(k), \qquad x(k_0) = x_0.$$

Internal stability concerns boundedness and asymptotic behavior of

$$x(k) = \Phi(k, k_0)x_0, \qquad k \geq k_0,$$

Definition 22 (Uniform Stability) The discrete-time system $x(k+1) = A(k)x(k)$ is called uniformly stable if there exists a finite constant $\gamma > 0$ such that, for any k_0 and x_0,

$$\|x(k)\| \leq \gamma \|x_0\|, \qquad \forall k \geq k_0.$$

Equivalently, $\|\Phi(k, k_0)\| \leq \gamma$, $\forall k \geq k_0$.

"Uniform" means that the same bound γ works for every initial time k_0.

Uniform Asymptotic Stability

Definition 23 (Uniform Asymptotic Stability) The discrete-time linear state equation is called uniformly asymptotically stable if it is uniformly stable and, for every $\delta > 0$, there exists a positive integer $K(\delta)$ such that

$$\|x(k)\| \leq \delta \|x_0\|, \qquad \forall k \geq k_0 + K(\delta),$$

for every k_0 and x_0.

- The elapsed time $K(\delta)$ is independent of k_0.
- Equivalently,

$$\|\Phi(k, k_0)\| \leq \delta, \qquad k \geq k_0 + K(\delta).$$

- Pointwise convergence $\Phi(k, k_0) \to 0$ for every fixed k_0 is generally not sufficient.

Uniform Exponential Stability

Definition 24 (Uniform Exponential Stability) The discrete-time system is called uniformly exponentially stable if there exist constants $\gamma \geq 1$ and $0 \leq \lambda < 1$ such that

$$\|x(k)\| \leq \gamma \lambda^{k-k_0} \|x_0\|, \qquad \forall k \geq k_0.$$

Equivalently,

$$\|\Phi(k, k_0)\| \leq \gamma \lambda^{k-k_0}, \qquad \forall k \geq k_0.$$

Theorem 25 *For discrete-time linear systems,*

$$\text{UAS} \quad \Longleftrightarrow \quad \text{UES}.$$

Continuous-time analogy

$$e^{-\alpha(t-t_0)} \qquad \longleftrightarrow \qquad \lambda^{k-k_0}, \quad 0 \leq \lambda < 1.$$

Quadratic Lyapunov Functions for DT Systems

- Consider a time-dependent quadratic function

$$V(x,k) = x^\top P(k) x,$$

where $P(k) = P^\top(k) \succ 0$.

- Along the solutions of $x(k+1) = A(k)x(k)$, its first difference is

$$\begin{aligned}\Delta V(k) &:= V(x(k+1), k+1) - V(x(k), k) \\ &= x^\top(k+1)P(k+1)x(k+1) - x^\top(k)P(k)x(k) \\ &= x^\top(k)\left[A^\top(k)P(k+1)A(k) - P(k)\right]x(k).\end{aligned}$$

The derivative $\dot{V}$ in continuous time is replaced by the first difference ΔV in discrete time.

Lyapunov Criterion for Uniform Stability

Theorem 26 *Consider* $x(k+1) = A(k)x(k)$*. Suppose* $\exists$ *finite constants* $\eta, \rho > 0$ *and a symmetric matrix sequence* $P(k)$ *s.t.*

$$\left.\begin{aligned} \eta I \preceq P(k) \preceq \rho I, \\ A^\top(k)P(k+1)A(k) - P(k) \preceq 0, \end{aligned}\right\} \quad \forall k.$$

Then the system is uniformly stable.
It is uniformly exponentially stable if and only if $\exists$ *finite constants* $\eta, \rho, \nu > 0$ *and a symmetric matrix sequence* $P(k)$ *s.t.*

$$\left.\begin{aligned} \eta I \preceq P(k) \preceq \rho I, \\ A^\top(k)P(k+1)A(k) - P(k) \preceq -\nu I, \end{aligned}\right\} \quad \forall k.$$

Discrete-Time LTI Systems

For the LTI system

$$x(k+1) = Ax(k),$$

the solution is $x(k) = A^{k-k_0}x_0$.

Theorem 27 *(Schur Stability) The system is exponentially stable if and only if*

$$|\lambda_i(A)| < 1, \qquad i = 1, \ldots, n.$$

A matrix whose eigenvalues lie strictly inside the unit disk is called a Schur matrix or a Schur-stable matrix.

Continuous time:	$\mathrm{Re}[\lambda_i(A)] < 0,$
Discrete time:	$\lvert\lambda_i(A)\rvert < 1.$

Uniform Stability of Discrete-Time LTI Systems

Theorem 28 *The system $x(k+1) = Ax(k)$ is uniformly stable if and only if*

1. *every eigenvalue of A satisfies*

$$|\lambda_i(A)| \leq 1;$$

2. *every eigenvalue on the unit circle is semisimple, i.e., every corresponding Jordan block has size one.*

Example

For $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, we have $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, which is unbounded. Hence the system is unstable even though both eigenvalues satisfy $|\lambda_i(A)| = 1$.

Continuous- and Discrete-Time Comparison

	Continuous time	Discrete time
System	$\dot{x} = A(t)x$	$x(k+1) = A(k)x(k)$
Lyapunov function	$x^\top P(t)x$	$x^\top P(k)x$
Variation	$\dot{V}$	ΔV
Lyapunov operator	$A^\top P + PA + \dot{P}$	$A^\top(k)P(k+1)A(k) - P(k)$
LTI equation	$A^\top P + PA = -M$	$A^\top PA - P = -M$
LTI stability region	$\mathrm{Re}[\lambda_i(A)] < 0$	$\lvert\lambda_i(A)\rvert < 1$

8. Reading 3: External/BIBO Stability ($\star$)

BIBO Stability

External stability studies the input-output map under zero initial conditions:

$$\dot{x} = A(t)x + B(t)u, \qquad y = C(t)x + D(t)u.$$

Definition 29 (BIBO Stability) The system is bounded-input bounded-output stable if every bounded input produces a bounded output:

$$\sup_{t \geq t_0} \|u(t)\| < \infty \quad \Longrightarrow \quad \sup_{t \geq t_0} \|y(t)\| < \infty.$$

It is uniformly BIBO stable if $\exists k > 0$, independent of t_0, s.t.

$$x(t_0) = 0, \quad \|u(t)\| \leq 1 \quad \Longrightarrow \quad \|y(t)\| \leq k, \quad \forall t \geq t_0.$$

Key point: internal stability concerns the zero-input state; external stability concerns the forced input-output behavior.

LTV Systems: Impulse Response Criterion

For the LTV system with $D(t) = 0$, $\dot{x} = A(t)x + B(t)u,\ y = C(t)x$, the input-output response is

$$y(t) = \int_{t_0}^{t} H(t,\tau)u(\tau)d\tau,$$

where

$$H(t,\tau) = C(t)\Phi(t,\tau)B(\tau), \qquad t \geq \tau,$$

is the impulse response matrix.

A standard sufficient condition for uniform BIBO stability is

$$\int_{t_0}^{t} \|H(t,\tau)\| d\tau \leq L, \qquad \forall t \geq t_0.$$

Indeed, if $\|u(t)\| \leq 1$, then $\|y(t)\| \leq \int_{t_0}^{t} \|H(t,\tau)\| \|u(\tau)\| d\tau \leq L$. If $D(t) \neq 0$, also require $\sup_t \|D(t)\| < \infty$.

LTI Systems: A Simpler Condition

For the LTI system (A, B, C, D), $H(t) = Ce^{At}B,\ t \geq 0$. The system is BIBO stable if

$$\int_0^\infty \|Ce^{At}B\| dt < \infty \quad \|D\| < \infty$$

Equivalently, BIBO stability can be verified from the TF

$$G(s) = C(sI - A)^{-1}B + D:$$

BIBO stability $\iff$ all poles of $G(s)$ lie in $\mathbb{C}_{<0}$.

Note that poles are considered after pole-zero cancellations. Therefore, for a nonminimal realization, $G(s)$ may be BIBO stable even when A is not Hurwitz.

Internal vs External Stability

For LTI systems,

$$\text{internal stability} \quad \Longrightarrow \quad \text{BIBO stability.}$$

The converse is not always true.

Example 30 Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0.$$

The TF is $G(s) = C(sI - A)^{-1}B + D = 1/(s+1)$. Since the only pole of $G(s)$ is -1, the input-output map is BIBO stable. However, A has an unstable eigenvalue $\lambda = 1$, so the realization is not internally stable.

This is related to minimality, which will be learnt at the end of the course.

Sources and Further Reading

- Stability concepts and state-transition criteria: They draw primarily on Rugh and (Antsaklis and Michel).
- Stability of LTI systems: They follow standard treatments in Rugh, (Callier and Desoer), and (Antsaklis and Michel).
- Lyapunov's direct method: The quadratic Lyapunov conditions for LTV and LTI systems, converse Lyapunov results, and the algebraic Lyapunov equation draw mainly on Rugh and Khalil.
- Matrix-measure criteria: The logarithmic-norm bounds and the associated sufficient conditions for stability are based on Vidyasagar.[1]
- Discrete-time and input–output stability: The material on Schur stability, discrete-time Lyapunov equations, BIBO stability, impulse-response criteria, and transfer-function pole conditions draws on Rugh, (Callier and Desoer), and (Antsaklis and Michel).

The historical remarks on Lyapunov refer to his original memoir on the stability of motion and its subsequent French translation. Selected proofs, examples, and comparisons were reorganized by the instructor.

[1] M. Vidyasagar, *Nonlinear Systems Analysis*, 2nd ed., Prentice Hall, 1993.