

# ELE6202E - Multivariable Systems

## Lecture 5: Controllability, Reachability, and Observability

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# **1. Motivation and Definitions**

# Motivation

In this chapter, we

- ▶ are interested in the controlled system

$$\dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0 \quad (\text{CT})$$

or in discrete time

$$x_{k+1} = A_k x_k + B_k u_k, \quad x_{k_0} = x_0 \quad (\text{DT})$$

- ▶ and want to understand how the input  $u$  can influence the state trajectory  $x(t)$  or  $x_k$ .

Can we use  $u(\cdot)$  to change the state from any  $x_0$  at time  $t_0$  (or  $k_0$ ) to any  $x_f$  at time  $t_f$  for  $t_f < \infty$  or  $t \rightarrow \infty$ ? This is also useful to start addressing the issue of synthesising feedback controllers.

## Motivating Examples

**Example 1** The system

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

satisfies  $x_2(t) = e^{-t}x_2(0)$ . This is **not** influenced by  $u$ .

**Example 2** Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u.$$

If the initial condition satisfies  $x_1(0) = x_2(0)$ , then  $x_1(t) \equiv x_2(t)$  for all  $t \geq t_0$ . We **cannot** drive the state to  $x_f$  with  $x_1(t_f) \neq x_2(t_f)$ .

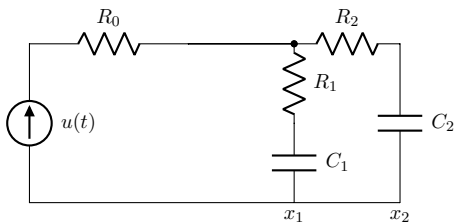
### Example 3: Electric Circuit

In the following circuit,  $R_1 = R_2 = R_3 = 10k\Omega$ ,  $C_1 = C_2 = 33\mu\text{F}$ , and  $x_1, x_2$  represent the voltages. The dynamics is given by

$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For zero initial condition, the solution is

$$x(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \int_0^t e^{-(t-s)} u(s) ds \implies x(t_f) \in \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

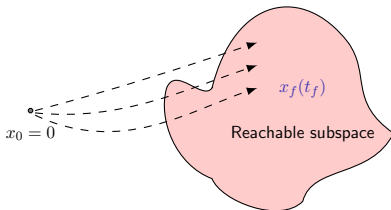


## Reachability and Reachable Subspace

**Definition 3** Given  $t_0 < t_f$ , the **reachable subspace** of the system  $\dot{x} = A(t)x + B(t)u$  on  $[t_0, t_f]$ , is the set of state  $x_f$  for which  $\exists$  an input  $u(\cdot)$  on  $[t_0, t_f]$  transferring the state **from**  $x_0 = 0$  at  $t_0$  to  $x_f$  at  $t_f$ . I.e.

$$\mathcal{R}(t_0, t_f) := \left\{ x_f \in \mathbb{R}^n : \forall u(\cdot), x_f = \int_{t_0}^{t_f} \Phi(t_f, s) B(s) u(s) ds \right\}.$$

The system (or the pair  $(A(\cdot), B(\cdot))$ ) is (completely) **reachable** on  $[t_0, t_f]$  if  $\mathcal{R}(t_0, t_f) = \mathbb{R}^n$ .



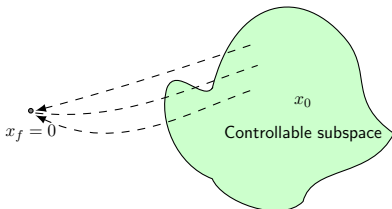
## Controllability and Controllable Subspace

**Definition 4** Given  $t_0 < t_f$ , the **controllable subspace** of the system  $\dot{x} = A(t)x + B(t)u$  on  $[t_0, t_f]$  is the set of states  $x_0$  for which  $\exists$  an input  $u(\cdot)$  on  $[t_0, t_f]$  transferring the state from  $x_0$  at  $t_0$  to  $x(t_f) = 0$ . I.e.

$$\mathcal{C}[t_0, t_f] = \left\{ x_0 \in \mathbb{R}^n : \exists u \text{ s.t. } \Phi(t_f, t_0)x_0 + \int_{t_0}^{t_f} \Phi(t_f, s)B(s)u(s)ds = 0 \right\}.$$

The system (or the pair  $(A(\cdot), B(\cdot))$ ) is (completely) **controllable** on  $[t_0, t_1]$  if

$$\mathcal{C}[t_0, t_1] = \mathbb{R}^n.$$



## Remarks

- ▶ In general, a state can be controllable but not reachable.
- ▶ For **continuous-time** linear systems,

controllability  $\iff$  reachability.

This means  $\exists$  a  $u(\cdot)$  to transfer any  $x_0$  to any  $x_f$  in a finite time.

- ▶  $\mathcal{R}(t_0, t_f)$  and  $\mathcal{C}(t_0, t_1)$  are indeed linear subspaces of  $\mathbb{R}^n$ , because they are the **images** of the linear operators.
- ▶ For discrete-time systems:

$$\mathcal{R}(k_0, k_f) = \left\{ x_f \in \mathbb{R}^n : \forall u, \text{ s.t. } x_f = \sum_{i=k_0}^{k_f-1} \Phi(k_f, i+1)B[i]u[i] \right\}$$

$$\mathcal{C}(k_0, k_f) = \left\{ x_0 \in \mathbb{R}^n : \exists u \text{ s.t. } \Phi(k_f, k_0)x_0 = - \sum_{i=k_0}^{k_f-1} \Phi(k_f, i+1)B[i]u[i] \right\}$$

Note the non-invertibility of  $\Phi(\cdot, \cdot)$  for DT.

**Proposition 1** A dynamical system  $\mathcal{D} := (\mathcal{U}, \Sigma, \mathcal{Y}, s, r)$  is completely controllable on  $[t_0, t_1]$  if and only if for any  $x_0 \in \Sigma$ , the map

$$s(t_1, t_0, x_0, \cdot) : \mathcal{U} \rightarrow \Sigma$$

is surjective, i.e. it maps  $\mathcal{U}$  onto  $\Sigma$ .

► In our case, the mapping  $s$  satisfies

$$x_1 = s(t_1, t_0, x_0, u_{[t_0, t_1]}(\cdot)) = \Phi(t_1, t_0)x_0 + \underbrace{\int_{t_0}^{t_1} \Phi(t_1, s)B(s)u(s)ds}_{:= \mathcal{L}_c(u_{[t_0, t_1]}(\cdot))},$$

where the linear map

$$\mathcal{L}_c : PC^m[t_0, t_1] \rightarrow \mathbb{R}^n : u(\cdot) \rightarrow \int_{t_0}^{t_1} \Phi(t_1, s)B(s)u(s)ds$$

is called **reachability map**.

## 2. Adjoint Map

## Motivation

- ▶ Recall that

$$x(t_1) = \Phi(t_1, t_0)x(t_0) + \mathcal{L}_c(u_{[t_0, t_1]}(\cdot)).$$

From early discussion, we have

$$\text{CC on } [t_0, t_1] \iff \text{Im}(\mathcal{L}_c) = \mathbb{R}^n.$$

- ▶ We are familiar with the case of static maps

### Finite-dimensional Least Squares Estimation

Consider a linear squares problem

$$A^\top u = b, \quad A = [a_1 \mid \dots \mid a_n] \in \mathbb{R}^{d \times n} \quad (1)$$

with  $b \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^d$  and  $d > n$ .

This is the **under constrained case** of a linear regressor, where we know a few projections of  $u$  of  $\# \text{dim} > n$ .

The system  $A^T u = b$  always has a solution, i.e.  $\text{Im}(A^T) = \mathbb{R}^n$  if and only if the columns of  $A$  are linearly independent:

$$\text{Im}(A^T) = \mathbb{R}^n \iff \text{Im}(A^T A) = \mathbb{R}^n \iff \text{Full rank } A^T A.$$

Then, all solutions are

$$u = \underbrace{A(A^T A)^{-1}b}_{u_0} + \text{Ker}(A^T).$$

The particular solution  $u_0$  minimizes the cost function  $\|u\|^2 = \langle u, u \rangle$ .

### Questions:

- ▶ Do we have  $\text{Im}(\mathcal{L}_c) = \mathbb{R}^n \iff \text{Im}(\mathcal{L}_c \mathcal{L}_c^*) = \mathbb{R}^n$ ?
- ▶ How can we define  $\mathcal{L}_c^*$  for  $\mathcal{L}_c : PC^m[t_0, t_1] \rightarrow \mathbb{R}^n$ ?

## Adjoint Map

**Definition 5** (Adjoint map) Let  $\mathcal{A} : U \rightarrow V$  be a bounded linear map between Hilbert spaces. Its adjoint  $\mathcal{A}^* : V \rightarrow U$  is defined by

$$\langle v, \mathcal{A}u \rangle_V = \langle \mathcal{A}^*v, u \rangle_U, \quad \forall v \in V, \forall u \in U.$$

In finite-dimensional real spaces, this is the transpose, i.e.  $A^* = A^\top$ .

**Proposition 2** *The adjoint exists and is unique.*

**Proposition 3** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be both linear maps from  $U$  to  $V$ :*

1.  $(\mathcal{A} + \mathcal{B})^* = \mathcal{A}^* + \mathcal{B}^*$ ;
2.  $(\alpha\mathcal{A})^* = \bar{\alpha}\mathcal{A}^*$ ;
3. *If  $U = V$ ,  $(\mathcal{A}\mathcal{B})^* = \mathcal{B}^*\mathcal{A}^*$ ;*
4.  $(\mathcal{A}^*)^* = \mathcal{A}$ .

**Definition 6** (Orthogonal direct sum) Let  $(V, \mathbb{F}, \langle \cdot, \cdot \rangle)$  be a Hilbert space. Recall that for a subspace  $M \subset V$ , its orthogonal complement is defined as

$$M^\perp := \{y \in V \mid \langle x, y \rangle = 0, \forall x \in M\}.$$

We then have the *orthogonal direct sum* as  $V = M \oplus M^\perp$ , i.e.,

$$\forall x \in V, \exists! y \in M \text{ such that } x - y \in M^\perp.$$

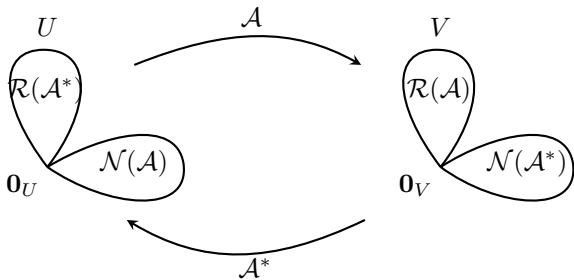
We also use

$$V = M \overset{\perp}{\oplus} M^\perp$$

to denote it.

**Theorem 7** Consider a continuous linear map  $\mathcal{A} : U \rightarrow V$ . Then

- ①  $V = \mathcal{R}(\mathcal{A}) \overset{\perp}{\oplus} \mathcal{N}(\mathcal{A}^*);$
- ②  $U = \mathcal{R}(\mathcal{A}^*) \overset{\perp}{\oplus} \mathcal{N}(\mathcal{A});$
- ③  $\mathcal{N}(\mathcal{A}\mathcal{A}^*) = \mathcal{N}(\mathcal{A}^*);$
- ④  $\mathcal{R}(\mathcal{A}\mathcal{A}^*) = \mathcal{R}(\mathcal{A});$
- ⑤  $\mathcal{N}(\mathcal{A}^*\mathcal{A}) = \mathcal{N}(\mathcal{A});$
- ⑥  $\mathcal{R}(\mathcal{A}^*\mathcal{A}) = \mathcal{R}(\mathcal{A}^*).$



## Solution to $y = \mathcal{A}x$

Given a continuous linear map

$$\mathcal{A} : U \rightarrow V,$$

we want to solve the equation

$$y = \mathcal{A}x.$$

If  $\mathcal{A}$  is bijective, then  $\mathcal{A}^{-1}$  exists and  $x = \mathcal{A}^{-1}y$ ; if  $\mathcal{A}$  is not invertible, we consider three cases:

- 1  $\mathcal{A}$  is surjective;
- 2  $\mathcal{A}$  is injective;
- 3  $\mathcal{A}$  is neither surjective nor injective.

## Case 1: $\mathcal{A}$ is surjective

If  $\mathcal{A}$  is surjective,  $\forall y, \exists x \in U$  such that  $y = \mathcal{A}x$ . We want to find  $x$  with minimum norm, i.e.,

$$\min \|x\|^2, \quad \text{s.t. } y = \mathcal{A}x.$$

Let  $x = x_1 + x_2 \in U$ , where  $x_1 \in \mathcal{R}(\mathcal{A}^*)$  and  $x_2 \in \mathcal{N}(\mathcal{A})$  (why?).  
Then

$$y = \mathcal{A}x = \mathcal{A}(x_1 + x_2) = \mathcal{A}x_1,$$

and

$$\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 \geq \|x_1\|^2.$$

To minimize  $\|x\|^2$ , we can choose  $x_2 = \mathbf{0}_U$ , and  $x = x_1 = \mathcal{A}^*v_0$  for some  $v_0 \in V$ . Now

$$y = \mathcal{A}x = \mathcal{A}\mathcal{A}^*v_0.$$

To find  $v_0$ , we have

$$\begin{aligned}\mathcal{A} \text{ is surjective} &\implies \mathcal{R}(\mathcal{A}) = V \\ &\implies \mathcal{N}(\mathcal{A}\mathcal{A}^*) = \mathcal{N}(\mathcal{A}^*) = \{\mathbf{0}_V\} \\ &\implies \mathcal{A}\mathcal{A}^* \text{ is invertible.}\end{aligned}$$

Thus there exists a unique  $v_0 = (\mathcal{A}\mathcal{A}^*)^{-1}y$  such that

$$x = x_1 = \mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}y.$$

Note that for a matrix  $A$  with full row rank,  $A^*(AA^*)^{-1}$  is its pseudoinverse (or Moore–Penrose pseudoinverse).

Cases 2 and 3 are left for homework.

### **3. Controllability and Reachability Gramians**

- ▶ Recall that  $x(t_1) = \Phi(t_1, t_0)x(t_0) + \mathcal{L}_c(u_{[t_0, t_1]}(\cdot))$ . From early discussion, we have

$$\text{CC on } [t_0, t_1] \iff \text{Im}(\mathcal{L}_c) = \mathbb{R}^n \iff \text{Im}(\mathcal{L}_c \mathcal{L}_c^*) = \mathbb{R}^n.$$

- ▶ For any  $f(\cdot), g(\cdot) \in PC^m[t_0, t_1]$ :  $\langle f, g \rangle := \int_{t_0}^{t_1} f^\dagger(\tau)g(\tau)d\tau$ . For any  $x \in \mathbb{R}^n$ , we have

$$\begin{aligned} \langle x, \mathcal{L}_c u \rangle_{\mathbb{R}^n} &= x^\dagger \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau) d\tau \\ &= \int_{t_0}^{t_1} \left( B^\dagger(\tau) \Phi^\dagger(t_1, \tau) x \right)^\dagger u(\tau) d\tau \\ &= \left\langle B^\dagger(\cdot) \Phi^\dagger(t_1, \cdot) x, u(\cdot) \right\rangle_{PC^m[t_0, t_1]} \\ &= \langle \mathcal{L}_c^* x, u(\cdot) \rangle_{PC^m[t_0, t_1]}. \end{aligned}$$

Therefore,

$$\mathcal{L}_c^* : \mathbb{R}^n \rightarrow PC^m[t_0, t_1] : (\mathcal{L}_c^* x)(\cdot) = B^\dagger(\cdot) \Phi^\dagger(t_1, \cdot) x.$$

- ▶  $\mathcal{L}_c \mathcal{L}_c^*$  is represented by

$$\mathcal{L}_c \mathcal{L}_c^* : \mathbb{R}^n \rightarrow \mathbb{R}^n$$
$$x_0 \mapsto \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) B^\dagger(\tau) \Phi^\dagger(t_1, \tau) d\tau x_0.$$

- ▶ Define the *Reachability Gramian* as

$$W_R[t_0, t_1] := \mathcal{L}_c \mathcal{L}_c^* = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) B^\dagger(\tau) \Phi^\dagger(t_1, \tau) d\tau \in \mathbb{R}^{n \times n}.$$

- ▶ It is a symmetric and positive semidefinite matrix. We have

|   |
|---|
| CC on $[t_0, t_1]$ $\iff$ rank $W_R[t_0, t_1] = n.$ |
|---|

## Solution to Input $u(\cdot)$

We want to find input  $u(\cdot)$  steering from  $x_0$  to  $x_1$  with the least  $L_2$  norm, namely,

$$\langle u, u \rangle = \int_{t_0}^{t_1} u^\dagger(\tau)u(\tau) d\tau = \|u\|_2^2.$$

From Slide 17, we know that

$$\begin{aligned} u(\cdot) &= \mathcal{L}_c^* (\mathcal{L}_c \mathcal{L}_c^*)^{-1} (x_1 - \Phi(t_1, t_0)x_0) \\ &= \mathcal{L}_c^* W_R^{-1} (x_1 - \Phi(t_1, t_0)x_0) \\ &= B^\dagger(\cdot) \Phi^\dagger(t_1, \cdot) W_R^{-1} (x_1 - \Phi(t_1, t_0)x_0). \end{aligned}$$

## Controllability Gramian Condition

**Theorem 8** The pair  $(A(t), B(t))$  is completely controllable on  $[t_0, t_1]$  if and only if the controllability Gramian

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, s) B(s) B(s)^\top \Phi(t_0, s)^\top ds \succeq \alpha I$$

for some scalar  $\alpha > 0$ .

### Relation between Controllability and Reachability Gramians

$$\begin{aligned} W_R(t_0, t_1) &:= \int_{t_0}^{t_1} \Phi(t_1, s) B(s) B(s)^\top \Phi(t_1, s)^\top ds \\ &= \Phi(t_1, t_0) W_c(t_0, t_1) \Phi(t_1, t_0)^\top. \end{aligned}$$

## Alternative Proof (★)

We provide the following alternative proof without using adjoint maps.

( $\Leftarrow$ ) We need to show the existence of a control  $u(t)$  such that for any IC  $x_0$  the state can be driven to the origin at a given moment  $t = t_1$ . We select the particular one

$$u(t) = -B^\top(t)\Phi^\top(t_0, t)W^{-1}(t_0, t_1)x_0.$$

Then,

$$\begin{aligned}x(t_1) &= \Phi(t_1, t_0)x_0 - \int_{t_0}^{t_1} \Phi(t_1, s)B(s)B(s)^\top \Phi(t_0, s)^\top ds W^{-1}(t_0, t_1)x_0 \\ &= \Phi(t_1, t_0)x_0 - \Phi(t_1, t_0) \int_{t_0}^{t_1} \Phi(t_0, s)B(s)B(s)^\top \Phi(t_0, s)^\top ds W^{-1}(t_0, t_1)x_0 \\ &= \Phi(t_1, t_0)x_0 - \Phi(t_1, t_0)x_0 \\ &= 0.\end{aligned}$$

( $\Rightarrow$ ) We need to show a contradiction by assuming the controllability and the non-invertibility of  $W(t_0, t_1)$ . First, we have  $\exists x_a \neq 0$  s.t.  $W(t_0, t_1)x_a = 0$ , then

$$\begin{aligned}x_a^\top W(t_0, t_1)x_a &= \int_{t_0}^{t_1} \|B^\top(t)\Phi^\top(t_0, t)x_a\|^2 dt = 0 \\ \Rightarrow B^\top(t)\Phi^\top(t_0, t)x_a &= 0, \quad \forall t \in [t_0, t_1].\end{aligned}$$

From the complete controllability,  $\exists$  an input  $u(t)$  that transfers  $x_a$  to 0 in  $[t_0, t_1]$ , i.e.

$$0 = \Phi(t_1, t_0)x_a + \int_{t_0}^{t_1} \Phi(t_1, s)B(s)u(s)ds.$$

Then,

$$x_a^\top x_a = - \int_{t_0}^{t_1} x_a^\top \Phi(t_0, s)B(s)u(s)ds = 0,$$

which means  $x_a = 0$ . This leads to a contradiction. ■

## Remarks

- ▶  $W(t_0, t_1)$  is symmetric and semi-definite positive.
- ▶ The function  $v^\top W(t_0, t)v$  for non-zero vector  $v \in \mathbb{R}^n$  is a non-decreasing function of  $t$ , since

$$W(t_0, t_2) \succeq W(t_0, t_1), \quad \forall t_1 \leq t_2.$$

Intuitively, the system becomes “more controllable/reachable” as increasing time.

- ▶ The resulting control  $u(t)$  is an **open-loop** input, not usually practical for control engineering.
- ▶ Controllability and reachability are **finite-time** concepts. However, there are uncontrollable systems for which the origin can be reached in infinite time from any state, e.g.

$$\dot{x} = -x + 0 \cdot u.$$

This system is uncontrollable.

**Theorem 9** (*Controllable and Reachable Subspaces*) Given two times  $t_1 \geq t_0 \geq 0$ ,

$$\mathcal{C}(t_0, t_1) = \text{Im } W_c(t_0, t_1), \quad \mathcal{R}(t_0, t_1) = \text{Im } W_R(t_0, t_1).$$

Moreover,

- ▶ if  $x_0 = W_c(t_0, t_1)\eta_0 \in \text{Im } W_c(t_0, t_1)$ , the control

$$u(t) = -B(t)^\top \Phi(t_0, t)^\top \eta_0, \quad \forall t \in [t_0, t_1]$$

can be used to transfer state from  $x(t_0) = x_0$  to  $x(t_1) = 0$ .

- ▶ if  $x_1 = W_R(t_0, t_1)\eta_1 \in \text{Im } W_R(t_0, t_1)$ , the control

$$u(t) = B(t)^\top \Phi(t_1, t)^\top \eta_1, \quad t \in [t_0, t_1]$$

can be used to transfer state from  $x(t_0) = 0$  to  $x(t_1) = x_1$ .

The proof is left for homework.

## Open-Loop Minimum-Energy Control ★

Consider the case  $W_c(t_0, t_1) \succ 0$ . We know the open-loop control

$$u(t) = -B^\top(t)\Phi^\top(t_0, t)\eta_0, \quad \eta_0 := W^{-1}(t_0, t_1)x_0.$$

can transfer  $x(t_0) = x_0$  to  $x(t_1) = 0$ . There may be other controls that achieve the same goal.

If there is another control  $\tilde{u}(\cdot)$ , we should have

$$x_0 = - \int_{t_0}^{t_1} \Phi(t_0, s)B(s)u(s)ds = - \int_{t_0}^{t_1} \Phi(t_0, s)B(s)\tilde{u}(s)ds = 0,$$

Therefore,

$$\int_{t_0}^{t_1} \Phi(t_0, s)B(s)v(s)ds = 0 \text{ with } v := \tilde{u} - u. \quad (2)$$

$$\begin{aligned}
\int_{t_0}^{t_1} \|\tilde{u}(s)\|^2 ds &= \int_{t_0}^{t_1} \left\| \underbrace{-B(s)^\top \Phi^\top(t_0, s)\eta_0}_{u(s)} + v(s) \right\|^2 ds \\
&= \eta_0^\top W_c(t_0, t_1)\eta_0 + \int_{t_0}^{t_1} \|v(s)\|^2 ds \\
&\quad - 2\eta_0^\top \int_{t_0}^{t_1} \Phi(t_0, s)B(s)v(s)ds.
\end{aligned}$$

The last term should be zero due to (2).

Therefore, the minimum energy is given by

$$\int_{t_0}^{t_1} \|\tilde{u}(s)\|^2 ds = x_0^\top W_c(t_0, t_1)^{-1}x_0.$$

## **4. Controllability for LTI Systems**

## Controllability for LTI Systems

In the LTI case

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0,$$

the controllability Gramian becomes

$$W_c(t_0, t_1) = \int_0^{t_1-t_0} e^{-At} B B^T e^{-A^T t} dt.$$

Note

1. For LTI systems, reachable and controllable subspaces coincide, so we focus on controllability.
2. These subspaces **do not** depend on time, and we do not need to specify  $[t_0, t_1]$ .
3. It is possible to reach the reachable subspace as fast as desired.

## Controllability Matrix

**Definition 10** For the pair  $(A, B)$ , define the controllability matrix

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] \in \mathbb{R}^{n \times nm}.$$

**Theorem 11** (*J.P. Hespanha, pp. 139*) For LTI systems,

$$\mathcal{R}[t_0, t_1] = \text{Im}(\mathbb{C}) = \text{Im}(W_c(t_0, t_1)) = \mathcal{C}[t_0, t_1]$$

### Proof

It remains only to show  $\text{Im}(\mathbb{C}) = \text{Im}(W_c)$ . Let  $x_a \in \mathcal{R}[t_0, t_1]$ , and thus  $x_a = \int_{t_0}^{t_1} e^{A(t_1-s)} B u(s) ds$  for some  $u(\cdot)$ . Using the Cayley-Hamilton theorem,  $\exists \alpha_1(\cdot), \dots, \alpha_n(\cdot)$ , s.t.  $e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i$ , for all  $t \in \mathbb{R}$ . Then,

## Proof.

$$x_a = \sum_{i=0}^{n-1} A^i B \int_{t_0}^{t_1} \alpha_i(t_1 - s) u(s) ds = \mathbb{C} \begin{bmatrix} \int_{t_0}^{t_1} \alpha_0(t_1 - s) u(s) ds \\ \vdots \\ \int_{t_0}^{t_1} \alpha_{n-1}(t_1 - s) u(s) ds \end{bmatrix}.$$

Therefore,  $x_a \in \text{Im}(\mathbb{C})$ .

Conversely, assume  $x_b \in \text{Im}(\mathbb{C})$ . We want to show  $x_b \in \text{Im}(W_c(t_0, t_1)) = \text{Ker}(W_c)^\perp$ . Let  $\eta \in \text{Ker} W_c$ , and

$$W_c \eta = 0 \implies \eta^\top e^{A(t_1-s)} B = 0, \forall s \in [t_0, t_1].$$

Take its  $i$ -th derivative  $d^i[\cdot]$ :  $(-1)^i \eta^\top A^i e^{A(t_1-s)} B = 0$  and fix  $s = t_1$ , thus

$$\eta^\top \mathbb{C} = 0 \implies \eta^\top x_b = 0.$$

We have  $x_b \in \text{Ker}(W_c)^\perp = \text{Im}(W_c)$ . ■

## Controllability Matrix Condition

**Theorem 12** The pair  $(A, B)$  is completely controllable if and only if

$$\text{rank } \mathbb{C} = n.$$

with the controllability matrix

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B].$$

Note: This is not true for LTV systems.

**Example 13** Consider the LTI pair  $(A, B)$  with

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The controllability matrix is

$$\mathbb{C} = [B \quad AB] = \begin{bmatrix} 1 & a_1 \\ 1 & a_2 \end{bmatrix}.$$

The system is controllable iff  $a_1 \neq a_2$ .

### Example 3: Electric Circuit (cont'd)

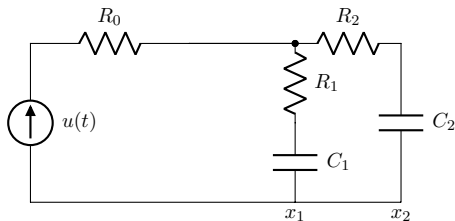
In the following circuit,  $R_1 = R_2 = R_3 = 10k\Omega$ ,  $C_1 = C_2 = 33\mu\text{F}$ , and  $x_1, x_2$  represent the voltages. The dynamics is given by

$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The controllability matrix is

$$C = [B \quad AB] = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

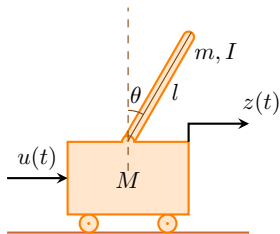
Hence,  $\text{rank}(C) = 1 < 2$ . Therefore, the system is not controllable.



## Example 4: Inverted Pendulum

The linearized model  $\dot{x} = Ax + Bu$  with  $x = \text{col}(z, \dot{z}, \theta, \dot{\theta})$  and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}.$$



The controllability matrix is

$$C = [B \quad AB \quad A^2B \quad A^3B] = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -10 \\ -2 & 0 & -10 & 0 \end{bmatrix}.$$

Since  $\text{rank}(C) = 4$ , the linearised inverted pendulum system is controllable.

## Computing Gramians

Consider the controllability Gramian

$$W_c(t, t_1) = \int_t^{t_1} \Phi(t, s) B(s) B(s)^\top \Phi(t, s)^\top ds.$$

Take the derivatives using Leibniz's rule: (Lyapunov ODE)

$$\frac{d}{dt} W_c(t, t_1) = -B(t) B(t)^\top + A(t) W_c(t, t_1) + W_c(t, t_1) A(t)^\top$$

with the terminal condition  $W_c(t_1, t_1) = 0$ .

## LTI Case

For a constant Hurwitz  $A$ ,  $W_c(\infty) = \int_0^\infty e^{At} B B^\top e^{A^\top t} ds$  is well defined, satisfying the Lyapunov equation

$$AW + WA^\top + BB^\top = 0. \quad (3)$$

**Theorem 14** If  $A$  is Hurwitz, then

$$\text{Controllability of } (A, B) \iff (3) \text{ has a p.d. solution.} \quad (4)$$

Hint: Consider the candidate  $W = \lim_{t \rightarrow \infty} W(t, t_1)$  on the previous slide. [Homework]

## **5. Discrete-Time Systems**

- ▶ In DT, reachability and controllability are not equivalent.

$$x_{k+1} = 0 \cdot x_k + 0 \cdot u_k$$

is controllable in one-time step, but not reachable.

- ▶ For  $x_{k+1} = A_k x_k + B_k u_k$ , the **reachable subspace** is

$$\mathcal{R}(k_0, k_f) = \left\{ x_f \in \mathbb{R}^n : \forall u, \text{ s.t. } x_f = \sum_{i=k_0}^{k_f-1} \Phi(k_f, i+1) B[i] u[i] \right\}.$$

- ▶ **Reachable Gramian** is defined as

$$W_R(k_0, k_f) = \sum_{j=k_0}^{k_f-1} \Phi(k_f, j+1) B(j) B^\top(j) \Phi^\top(k_f, j+1)$$

**Theorem 15** The DT system is reachable on  $[k_0, k_f]$  iff  $W_R(k_0, k_f)$  is invertible.

► Controllability Gramian

$$W_c(k_0, k_f) = \sum_{j=k_0}^{k_f-1} \Phi(k_0, j+1) B(j) B^\top(j) \Phi^\top(k_0, j+1),$$

requiring that we can define the backwards in time STM. ( $A_k$  are invertible.)

► Controllable Subspace:

$$\mathcal{C}(k_0, k_f) = \left\{ x_0 : \exists u \text{ s.t. } \Phi(k_f, k_0)x_0 = - \sum_{i=k_0}^{k_f-1} \Phi(k_f, i+1) B[i] u[i] \right\}$$

If  $W_c$  can be defined, then

$$\mathcal{C}(k_0, k_f) = \text{Im} (W_c(k_0, k_f)).$$

## Controllability for LTI Systems

For LTI systems  $x_{k+1} = Ax_k + Bu_k$ , we have

$$\begin{aligned}x_k &= A^k x_0 + \sum_{j=0}^{k-1} A^{k-1-j} B u_j \\ &= A^k x_0 + [A^{k-1}B \quad \dots \quad AB \quad B] \begin{bmatrix} u_0 \\ \vdots \\ u_{k-1} \end{bmatrix} \\ &:= A^k x_0 + \mathbf{R}_k \mathbf{u}_k.\end{aligned}$$

**Definition 16** We define the  $k$ -step reachability matrix

$$\mathbf{R}_k := [A^{k-1}B \quad \dots \quad AB \quad B].$$

- ▶ We call  $\mathcal{R}(0, n) = \text{Im}(\mathbf{R}_n)$  the reachable subspace.
- ▶ The LTI system is reachable iff  $\mathbb{R}_n$  has full rank.
- ▶ If a system is reachable, we can go from any  $x_0$  to any  $x_1$  in at most  $n$  steps, since

$$x_1 - A^n x_0 = \mathbf{R}_n \mathbf{u}$$

admits a solution  $\mathbf{u}$  due to full rank.

- ▶ If  $A$  is invertible, controllability and reachability are equivalent to each other.

## **6. Observability and Constructability**

## Observability and Constructability

Consider the **unforced** system

$$\begin{aligned}\dot{x} &= A(t)x, \quad x(t_0) = x_0 \\ y &= C(t)x.\end{aligned}\tag{5}$$

- ▶ Motivation: In many cases, the state  $x(t)$  cannot be measured directly. Can we estimate it?

$$(u, y) + \text{models} \rightarrow x(t)?$$

- ▶ Considering zero inputs does not entail loss of generality since the concept is unchanged in the presence of a known input. (only for linear systems)

Observability and controllability are dual concepts.

**Definition 17** (Observability and Constructability) The linear system

$$\dot{x} = A(t)x, \quad y = C(t)x \quad (6)$$

is **observable** on  $[t_0, t_f]$  if any initial state  $x_0$  is uniquely determined by the corresponding response  $y(t)$  and the input  $u(t)$  for  $t \in [t_0, t_f]$ .

The system is **constructable** on  $[t_0, t_f]$  if  $x(t_f)$  can be uniquely determined by  $u(t)$  and  $y(t)$  on  $[t_0, t_f]$ .

- ▶ Again in DT, we may have difficulties with constructibility when  $A_k$  is not invertible.
- ▶ It turns out to be more convenient to think in terms of unobservability/unconstructibility.

## Unobservable/Unconstructable Subspace

**Definition 18** (Unobservable/Unconstructable Subspace) Given  $t_1 \geq t_0 \geq 0$ , the **unobservable subspace** on  $[t_0, t_1]$  is

$$\bar{\mathcal{O}}[t_0, t_1] = \{x_0 \in \mathbb{R}^n : C(t)\Phi(t, t_0)x_0 = 0, \forall t \in [t_0, t_1]\}.$$

The **unconstructable subspace** on  $[t_0, t_1]$  is defined as

$$\bar{\mathcal{O}}_c[t_0, t_1] = \{x_1 \in \mathbb{R}^n : C(t)\Phi(t, t_1)x_1 = 0, \forall t \in [t_0, t_1]\}.$$

- ▶ For observable systems,  $\bar{\mathcal{O}}[t_0, t_1] = \{0\}$ . For constructable systems,  $\bar{\mathcal{O}}_c[t_0, t_1] = \{0\}$ .
- ▶ Define the operator

$$\mathcal{L}_o : \mathbb{R}^n \rightarrow PC^p[t_0, t_1]$$

$$x_o \mapsto C(t)\Phi(t, t_0)x_o.$$

Then,  $\bar{\mathcal{O}}[t_0, t_1] = \text{Ker } \mathcal{L}_o$ , thus being a linear subspace.

- ▶ An unobservable initial condition (i.e.  $x_0 \in \bar{\mathcal{O}}$ ) cannot be distinguished from the zero initial condition output:  $y \equiv 0$ .
- ▶ If  $\mathcal{L}_o[x_0] = y$ , then

$$\mathcal{L}_o[x_0 + \tilde{x}_0] = y, \quad \forall \tilde{x}_0 \in \text{Ker } \mathcal{L}_o.$$

So, it is also related to our ability to distinguish any two initial conditions from observing  $y$ .

$$\text{CO on } [t_0, t_1] \iff \text{Ker } (\mathcal{L}_o) = \{0_n\} \iff \text{Ker } (\mathcal{L}_o^* \mathcal{L}_o) = \{0_n\}.$$

## Adjoint Map $\mathcal{L}_o^*$

- ▶ For any  $y(\cdot) \in PC^p[t_0, t_1]$ , we have

$$\begin{aligned}\langle y, \mathcal{L}_o x_0 \rangle_{PC^p[t_0, t_1]} &= \int_{t_0}^{t_1} y^\dagger(\tau) C(\tau) \Phi(\tau, t_0) x_0 d\tau \\ &= \left\langle \int_{t_0}^{t_1} \Phi^\dagger(\tau, t_0) C^\dagger(\tau) y(\tau) d\tau, x_0 \right\rangle_{\mathbb{R}^n} \\ &= \langle \mathcal{L}_o^* y, x_0 \rangle_{\mathbb{R}^n}.\end{aligned}$$

- ▶ Therefore,

$$\mathcal{L}_o^* : PC^p[t_0, t_1] \rightarrow \mathbb{R}^n : \mathcal{L}_o^*(y(\cdot)) = \int_{t_0}^{t_1} \Phi^\dagger(\tau, t_0) C^\dagger(\tau) y(\tau) d\tau.$$

- ▶  $\mathcal{L}_o^* \mathcal{L}_o : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is represented by

$$\mathcal{L}_o^* \mathcal{L}_o : \mathbb{R}^n \rightarrow \mathbb{R}^n : x_0 \mapsto \int_{t_0}^{t_1} \Phi^\dagger(\tau, t_0) C^\dagger(\tau) C(\tau) \Phi(\tau, t_0) d\tau x_0.$$

## Observability Gramian

**Definition 19** (Observability Gramian) For the pair  $(C(t), A(t))$ , the observability Gramian on  $[t_0, t_f]$  is

$$W_o(t_0, t_f) = \int_{t_0}^{t_f} \Phi^\top(s, t_0) C^\top(s) C(s) \Phi(s, t_0) ds$$

**Theorem 20** (*Observability Gramian Condition*) The pair  $(C(t), A(t))$  is observable on  $[t_0, t_f]$  if and only if the observability Gramian  $W_o[t_0, t_f]$  is invertible. Besides,

$$\bar{O}[t_0, t_f] = \text{Ker } W_o[t_0, t_f].$$

## Proof

Multiplying the solution expression

$$y(t) = C(t)\Phi(t, t_0)x_0$$

on both sides by  $\Phi^\top(t, t_0)C^\top(t)$  and integrating yields

$$\int_{t_0}^{t_f} \Phi^\top(t, t_0)C^\top(t)y(t)dt = W_o[t_0, t_f]x_0$$

The LHS is determined by  $y(t), t \in [t_0, t_f]$ . Therefore, the above represents a linear algebraic equation for  $x_0$ . If  $M(t_0, t_f)$  is invertible, then  $x_0$  is uniquely determined.

On the other hand, if  $W_o$  is not invertible, then there exists a nonzero  $n \times 1$  vector  $x_a$  s.t.

$$W_o[t_0, t_f]x_a = 0.$$

This implies  $x_a^\top W_o[t_0, t_f]x_a = 0$  and it follows that

$$C(t)\Phi(t, t_0)x_a = 0, \quad t \in [t_0, t_f]$$

Thus  $x(t_0) = x_0 + x_a$  yields the same zero-input response for the system on  $[t_0, t_f]$  as  $x(t_0) = x_0$ , and the state equation fails to be observable on  $[t_0, t_f]$ .

The proof of  $\bar{\mathcal{O}}[t_0, t_f] = \text{Ker } W_o[t_0, t_f]$  is left for exercise. ■

## Observability for LTI Systems

- ▶ The observability Gramian  $W_o[t_0, t_f] \succeq 0$ , and  $\succ 0$  if and only if the system is observable on  $[t_0, t_f]$ .
- ▶ Numerical calculation via matrix differential equations.
- ▶ For LTI systems, it becomes  $W_o(t) = \int_0^t e^{A^\top s} C^\top C e^{As} ds$ . We define **observability matrix**

$$\mathbb{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

**Theorem 21** (*Observability Matrix Condition*) The LTI pair  $(C, A)$  with  $A \in \mathbb{R}^{n \times n}$  is observable if and only if

$$\text{rank } \mathbb{O} = n.$$

## **7. Duality of Controllability & Observability**

## Adjoint System

Consider the system

$$\begin{aligned} \dot{x} &= A(t)x, & x(t_0) &= x_0 \\ y &= C(t)x \end{aligned} \tag{7}$$

with the observability Gramian

$$W_o = \int_{t_0}^{t_1} \Phi(s, t_0)^\top C^\top(s) C(s) \Phi(s, t_0) ds$$

and its **adjoint system**

$$\dot{p} = -A^\top(t)p - C^\top(t)u, \quad p(t_1) = 0. \tag{8}$$

The STM of  $A(t)$  is simply denoted as  $\Phi(t, s)$ , and recall  $\Phi_{-A^\top}(t, s) = \Phi_A(s, t)^\top = \Phi(s, t)^\top$ . The controllability Gramian for (8) is

$$W_o(t_0, t_f) = \int_{t_0}^{t_1} \Phi_{-A^\top}(t_0, s) B(s) B^\top(s) \Phi_{-A^\top}(t_0, s)^\top ds = W_o$$

A system is observable iff the adjoint system is (backwards) reachable.

## LTI Systems

Duality allows transferring results from controllability to observability. Consider

$$\dot{x} = Ax, \quad y = Cx \quad (9)$$

and

$$\dot{x} = A^T x + C^T u. \quad (10)$$

By studying the Gramians, it is easy to see

$$\begin{aligned} \text{observability of } (A, C) &\iff \text{reachability of } (A^T, C^T) \\ \text{constructability of } (A, C) &\iff \text{controllability of } (A^T, C^T) \end{aligned} \quad (11)$$

## Sources and Further Reading

- ▶ **Controllability and reachability concepts:** They draw primarily on Rugh, (Callier and Desoer), and (Antsaklis and Michel).
- ▶ **Operator and adjoint-map formulation:** They follow the Hilbert-space treatment developed in (Callier and Desoer).
- ▶ **Gramians and minimum-energy control:** Rugh, (Callier and Desoer) and (Antsaklis and Michel).
- ▶ **Algebraic tests for LTI systems:** The controllability and observability matrices, their Cayley–Hamilton-based derivations, and the corresponding rank conditions follow standard treatments in Hespanha, Rugh, and (Antsaklis and Michel).
- ▶ **Observability and duality:** (Callier and Desoer) and (Antsaklis and Michel).

The circuit and inverted-pendulum examples, as well as selected proofs and operator-theoretic interpretations, were adapted and reorganized by the instructor.