ELE6202E - Multivariable Systems

Lecture 6: State Feedback

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Instructor: Bowen Yi

https://bwyi.github.io

POLYTECHNIQUE MONTRÉAL

Contents

1. Linear Feedback and Effects

Open-Loop System

A given system

$$\dot{x} = A(t)x + B(t)u,\ x(t_0) = x_0$$
$$y = C(t)x$$

is called plant or open-loop system.

- It may not behave appropriately in general (e.g. unstable, not fast, low precision, fragile).
- $\exists$ modeling uncertainties (e.g. uncertain parameters, linearization, neglected dynamics, and measurement noises).

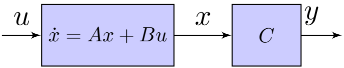


Figure 1: Open-loop

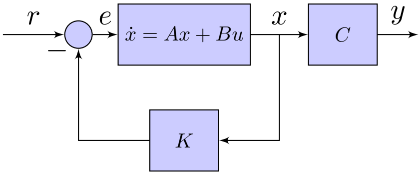


Figure 2: Closed-loop via Feedback

Static State Feedback

We consider a static state feedback

$$u(t) = K(t)x(t) + r(t), \tag{1}$$

with r a new input variable. The matrix K should be chosen such that the closed-loop system $r \mapsto x$:

$$\dot{x} = [A(t) + B(t)K(t)]x + B(t)r \tag{2}$$

has desired properties.

- We call it "static" since the input u depends on $r(t)$ and $x(t)$ at that same time.
- The closed loop is still a linear system.
- Example: Given an unstable $A(t)$, we select $K(t) \in \mathbb{R}^{m \times n}$ to make $(A + BK)$ stable. (Stabilization)

2. Controllability Decomposition

Decomposition into Controllable and Uncontrollable Parts

Theorem 1 Suppose the pair (A, B) satisfies

$$\text{rank}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = q$$

with $0 < q < n$. Then, $\exists$ an invertible $n \times n$ matrix P such that

$$P^{-1}AP = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0_{(n-q)\times q} & \hat{A}_{22} \end{bmatrix}, \quad P^{-1}B = \begin{bmatrix} \hat{B}_1 \\ 0_{(n-q)\times m} \end{bmatrix}$$

where $\hat{A}_{11} \in \mathbb{R}^{q\times q}$, $\hat{B}_1 \in \mathbb{R}^{q\times m}$, and $(\hat{A}_{11}, \hat{B}_1)$ is controllable.

Proof

The coordinate change $\tilde{x} = P^{-1}x$ makes

$$\dot{x} = Ax + Bu \;\rightarrow\; \dot{\tilde{x}} = \underbrace{P^{-1}AP}_{\hat{A}}\,\tilde{x} + \underbrace{P^{-1}B}_{\hat{B}}\,u. \tag{3}$$

1) **Construction of** P: Select q linearly independent vectors $p_1, \ldots, p_q$ from the controllability matrix. Then, $\exists p_{q+1}, \ldots, p_n \in \mathbb{R}^n$ s.t.

$$P = \begin{bmatrix} p_1 & \ldots & p_q & p_{q+1} & \ldots & p_n \end{bmatrix}$$

is invertible. Therefore, the j-th column of B, denoted as B_j, can be linearly represented by $p_1, \ldots, p_q$.

The definition

$$\hat{B} := P^{-1}B := \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}$$

yields $P\hat{B} = B$. Thus, B_j can be linearly represented by P. Since P is full rank, it is necessary that $\hat{B}_2 = 0$.

2) From $\hat{A} = P^{-1}AP$, we have

$$P\hat{A} = \begin{bmatrix} Ap_1 & \dots & Ap_n \end{bmatrix}, \quad \text{with } \hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}.$$

By decomposing $P := [P_1 \ P_2]$, we have

$$\begin{cases} P_1\hat{A}_{11} + P_2\hat{A}_{21} = AP_1 = \begin{bmatrix} Ap_1 & \dots & Ap_q \end{bmatrix} \\ P_1\hat{A}_{12} + P_2\hat{A}_{22} = AP_2 \end{cases}$$

Each column of A^kB $(k \geq 0)$ is a linear combination of $p_1, \dots, p_q$, then $Ap_1, \dots, Ap_q$ can be written as linear combinations of them. Hence,

$$\text{Im}\ (P_1\hat{A}_{11} + P_2\hat{A}_{21}) = \text{Im}\ (AP_1) = L\{p_1, \dots, p_q\} \implies \hat{A}_{21} = 0.$$

3) Multiply the matrix P^{-1},

$$
\begin{aligned}
&P^{-1}\left[\begin{array}{cccc} B & AB & \cdots & A^{n-1}B \end{array}\right] \\
&= \left[\begin{array}{cccc} P^{-1}B & P^{-1}AB & \cdots & P^{-1}A^{n-1}B \end{array}\right] \\
&= \left[\begin{array}{cccc} \hat{B}_1 & \hat{A}_{11}\hat{B}_1 & \cdots & \hat{A}_{11}^{n-1}\hat{B}_1 \\ 0 & 0 & \cdots & 0 \end{array}\right]
\end{aligned}
$$

The rank is preserved at each step, and applying again the Cayley-Hamilton theorem shows that

$$
\text{rank}\left[\begin{array}{cccc} \hat{B}_1 & \hat{A}_{11}\hat{B}_1 & \cdots & \hat{A}_{11}^{q-1}\hat{B}_1 \end{array}\right] = q.
$$

■

Remark

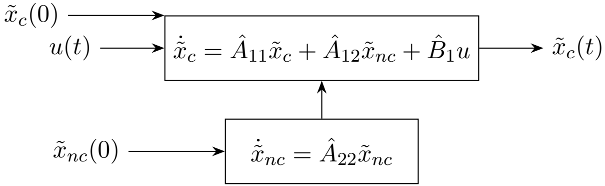
Writing the variable change as

$$\begin{bmatrix} \tilde{x}_c(t) \\ \tilde{x}_{nc}(t) \end{bmatrix} = P^{-1}x(t)$$

where the partition $\tilde{x}_c(t)$ is $q \times 1$, yields the decomposition

$$\begin{aligned} \dot{\tilde{x}}_c(t) &= \hat{A}_{11}\tilde{x}_c(t) + \hat{A}_{12}\tilde{x}_{nc}(t) + \hat{B}_1 u(t) \\ \dot{\tilde{x}}_{nc}(t) &= \hat{A}_{22}\tilde{x}_{nc}(t) \end{aligned}$$

$\tilde{x}_{nc}(t)$ is not influenced by the input, thus not controllable.



3. Popov-Belevitch-Hautus Test

PBH Test for Controllability

Theorem 2 *(PBH Test)* (A, B) is completely controllable if and only if rank $[A - \lambda I \quad B] = n$ for all eigenvalues λ of A.

Proof

The statement is equivalent to: (A, B) is not controllable $\iff$ $\text{rank}[A - \lambda I \quad B] < n$ for some eigenvalue λ of A.

($\Leftarrow$) Suppose $\text{rank}[A - \lambda I \quad B] < n$ for some eigenvalue λ, possibly complex. Thus, $\exists x \in \mathbb{C}^n$ s.t.

$$x^*[A - \lambda I \quad B] = 0$$

where x^* is complex conjugate transpose of x. This results in

$$x^* A = \lambda x^*, \; x^* B = 0$$

cont'd.

Then

$$x^*AB = \lambda x^*B = 0$$

and

$$x^*A^kB = \lambda^k x^*B = 0.$$

Thus, $x^*[B\ AB\ \ \ldots\ \ A^{n-1}] = 0$, and (A, B) is not controllable.

($\Rightarrow$) Assume (A, B) is not controllable. $\exists$ nonsingular V so that $\left(V^{-1}AV, V^{-1}B\right) = (\tilde{A}, \tilde{B})$, where

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix},\ \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix},$$

with $\tilde{A}_{11} \in \mathbb{R}^{q\times q}$ and $\tilde{A}_{22} \in \mathbb{R}^{n-q\times n-q}$. Now note that for any λ an eigenvalue of $\tilde{A}_{22}$,

$$\text{Rank}\left(\tilde{A}_{22} - \lambda I\right) < n - q$$

cont'd.

Next observe that

$$\begin{aligned} V^{-1}[A - \lambda I \quad B] \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} &= V^{-1}[AV - \lambda V \quad B] \\ &= [V^{-1}AV - \lambda I \quad V^{-1}B] \\ &= [\tilde{A} - \lambda I \quad \tilde{B}] \end{aligned}$$

Since V^{-1} and $\begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix}$ are both nonsingular,

$$\text{Rank}[\tilde{A} - \lambda I \quad \tilde{B}] = \text{Rank}[A - \lambda I \quad B] < n$$

for λ an eigenvalue of $\tilde{A}_{22}$. Finally, note that by the structure of $\tilde{A}$, an eigenvalue of $\tilde{A}_{22}$ is an eigenvalue of $\tilde{A}$. Since a change of basis does not change the eigenvalues, λ is also an eigenvalue of A. This concludes the proof that (A, B) not controllable implies there exists an eigenvalue of A, λ, such that rank$[A - \lambda I \quad B] < n$. ■

Corollary 3 (A, B) is completely controllable if and only if rank $[A - \lambda I \quad B] = n$ for $\forall \lambda \in \mathbb{C}$.

For $\lambda \notin \lambda(A)$, $(A - \lambda I)$ is full rank. The above always holds.

Controllability and Closed-Loop Stability

We will see that controllability of an LTI system is tightly related to our ability to place the eigenvalues of the closed-loop system matrix $(A + BK)$ as desired.

Theorem 4 Suppose the LTI pair (A, B) controllable. Let $\mu > 0$, then $\exists$ a state feedback controller $u = Kx$ s.t. the closed loop

$$\dot{x} = (A + BK)x$$

has all its eigenvalues λ_i satisfying

$$\mathrm{Re}(\lambda_i) \leq -\mu.$$

Note: The exact necessary and sufficient condition for assigning arbitrary closed-loop eigenvalues is controllability; stabilizability is enough if we only require the closed loop to be Hurwitz.

Proof.

Since (A, B) is controllable, the eigenvalue assignment theorem proved below allows us to place the eigenvalues of $A + BK$ at any conjugate-symmetric set of n complex numbers. Choose, for example,

$$\{-\mu - 1,\ -\mu - 2,\ \ldots,\ -\mu - n\}.$$

Then every closed-loop eigenvalue satisfies $\mathrm{Re}(\lambda_i) \leq -\mu$. ■

4. Canonical Forms for Controllable Systems

We need to find K such that the system matrix $A + BK$ has the desired eigenvalues. First, we consider the SISO case ($m = 1$).

Proposition 1 *There exists a nonsingular matrix* T *such that*

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ -\alpha_n & \cdots & \cdots & -\alpha_1 \end{bmatrix}, \text{ and } \tilde{b} = T^{-1}b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$\iff$ (A, b) *is completely controllable.* (4)

This form $(\tilde{A}, \tilde{b})$ is called the controllability Canonical Form.

Proof

($\Rightarrow$) If $\tilde{A}$ and $\tilde{b}$ are given as in Eq. (4), then

$$\tilde{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \tilde{A}\tilde{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -\alpha_1 \end{bmatrix}, \quad \tilde{A}^2\tilde{b} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ -\alpha_1 \\ \alpha_2 + \alpha_1^2 \end{bmatrix}, \quad \cdots$$

Therefore $[\tilde{b}\ \tilde{A}\tilde{b}\ \cdots\ \tilde{A}^{n-1}\tilde{b}]$ is full rank. Invoking coordinate changes do not change the controllability, and thus (A, b) is CC.

($\Leftarrow$) We prove by constructing T. Since we want $\tilde{b} = T^{-1}b = e_n$, we select $Te_n = b$.

$$\begin{aligned} \widetilde{A}e_n = e_{n-1} - \alpha_1 e_n \implies e_{n-1} &= \widetilde{A}e_n + \alpha_1 e_n \\ &= (T^{-1}AT)(T^{-1}b) + \alpha_1 T^{-1}b \\ &= T^{-1}(Ab + \alpha_1 b), \end{aligned}$$

and

$$\begin{aligned}\widetilde{A}e_{n-1} =& e_{n-2} - \alpha_2 e_n \implies \\ & e_{n-2} = \widetilde{A}e_{n-1} + \alpha_2 e_n = (T^{-1}AT)T^{-1}(Ab + \alpha_1 b) + \alpha_2 T^{-1}b \\ & \quad\quad = T^{-1}(A^2 b + \alpha_1 Ab + \alpha_2 b).\end{aligned}$$

Proceeding until $\widetilde{A}e_1$, we have

$$I = [e_1\, e_2\, \cdots\, e_n] = T^{-1}[b\, Ab\, \cdots\, A^{n-1}b]\begin{bmatrix}\alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & 1 \\ \alpha_{n-2} & \cdots & \alpha_1 & 1 & \\ \vdots & & \vdots & & \\ \alpha_1 & 1 & & & \\ 1 & & & & \end{bmatrix}$$

Because (A, b) is controllable and thus $[b\, Ab\, \cdots\, A^{n-1}b]$ is full rank, we can obtain T as a nonsingular matrix:

$$T = [b\, Ab\, \cdots\, A^{n-1}b] \begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & 1 \\ \alpha_{n-2} & \cdots & \alpha_1 & 1 & \\ \vdots & & \vdots & & \\ \alpha_1 & 1 & & & \\ 1 & & & & \end{bmatrix}.$$

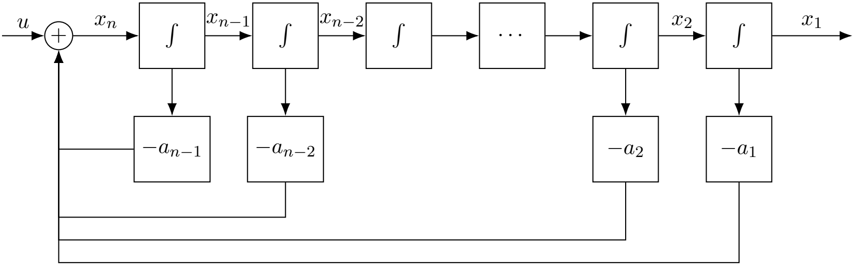
■

Note: The column vectors of T can be recursively defined by

$$\begin{cases} T_n = b \\ T_k = a_k b + AT_{k+1}, \; (k = 1, \ldots, n-1) \end{cases} \tag{5}$$

with $\det(sI - A) =: s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$.

Schematics of the Canonical Form



Example 1

Consider the pair (A, B) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

which is completely controllable but not in the canonical form.

Solution. The characteristic polynomial: $\det(sI - A) = s^3 - s$. Then

$$T_3 = b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; T_2 = 0 \cdot b + Ab = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \; T_1 = -1 \cdot b + AT_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

The matrix T is

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Theorem 5 *(Eigenvalue Assignment)* Assume (A, B) is controllable, with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Let $\Lambda := \{\lambda'_1, \ldots, \lambda'_n\}$ be any set of complex numbers (desired eigenvalues), s.t. $\forall \lambda' \in \Lambda$, $\lambda'^* \in \Lambda$ (complex conjugate pairs or real). Then, $\exists\, K \in \mathbb{R}^{m\times n}$ s.t.

$$\lambda(A + BK) = \Lambda.$$

For single input case $(m = 1)$, the target is equivalent to

$$\det(sI - (A + bK)) = \prod_{i=1}^{n}(s - \lambda'_i) = s^n + a'_{n-1}s^{n-1} + \ldots + a'_0.$$

The canonical form is

$$\tilde{A} + \tilde{b}\tilde{K} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a'_0 & \cdots & \cdots & -a'_{n-1} \end{bmatrix}$$

5. Eigenvalue Assignment and Stabilizability

Theorem 6 *(Eigenvalue Assignment)* Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, $\exists K \in \mathbb{R}^{m\times n}$ s.t. the n eigenvalues of $A + BK$ can be assigned to arbitrary, real or complex conjugate, locations iff (A, B) is controllable.

Proof

$(\Rightarrow)$ Suppose $\lambda(A + BK)$ have been arbitrarily assigned and that (A, B) is not controllable. This would lead to a contradiction. Since (A, B) is uncontrollable, $\exists Q$ that will separate the controllable part from the uncontrollable part.

$$\begin{aligned} Q^{-1}(A + BK)Q &= Q^{-1}AQ + (Q^{-1}B)(KQ) \\ &= \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} [K_1, K_2] \\ &= \begin{bmatrix} A_1 + B_1K_1 & A_{12} + B_1K_2 \\ 0 & A_2 \end{bmatrix}. \end{aligned}$$

where $[K_1, K_2] := KQ$ and (A_1, B_1) is controllable.

cont'd.

The coordinate transformation does not change eigenvalues, i.e.

$$\lambda(A + BK) = \lambda(Q^{-1}(A + BK)Q).$$

However, we cannot change the eigenvalues in the A_2-part. This contradicts to the arbitrarily assigned eigenvalues of $A + BK$.

($\Leftarrow$) Let (A, B) be fully controllable. Then by using any of the eigenvalue assignment algorithms presented later in this section, all the eigenvalues of $A + BK$ can be arbitrarily assigned. ■

Stabilizability

Definition 7 A pair (A, B) (or the system $\dot{x} = Ax + Bu$) is stabilizable if for every initial condition $x_0 \in \mathbb{R}^n$, there exists an input $u(\cdot)$ s.t.

$$\lim_{t \to \infty} X(t; t_0, x_0) = 0.$$

Corollary 8 The pair (A, B) is stabilizable if and only if it admits controllable decomposition

$$\tilde{A} = V^{-1}AV = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \tag{6}$$

has A_u either non-existence or Hurwitz.

- (A, B) is stabilizable iff the non-controllable modes are stable.
- Controllability $\implies$ stabilizability.

Example 2

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & -12 & 5 \end{bmatrix} \quad ; \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (A, b) \text{ cc and } A \text{ is unstable}$$

Using the classical convention $u = -kx$, if $\lambda_1^d = -1; \quad \lambda_{2,3}^d = -1 \pm j$, then $\chi_d(s) = (s - \lambda_1^d)(s - \lambda_2^d)(s - \lambda_3^d) = s^3 + 3s^2 + 4s + 2$. The question is $\exists k$ s.t. $\chi_{A-bk}(s) = \chi_d(s)$. Then,

$$\begin{aligned} A - b\tilde{k} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & -12 & 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \tilde{k}_1 & \tilde{k}_2 & \tilde{k}_3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 - \tilde{k}_1 & -12 - \tilde{k}_2 & 5 - \tilde{k}_3 \end{bmatrix} \end{aligned}$$

The desired closed-loop canonical form is obtained with $k = [14, -8, 8]^\top$.

6. Canonical Form for MIMO Systems

Controllability Index

To construct the corresponding controller form when we have multiple inputs $(m > 1)$ we need the following.

Definition 9 Let $B = [B_1 \ \dots \ B_m]$. For $j = 1, \dots, m$, the controllability index ρ_j is the smallest integer such that $A^{\rho_j} B_j$ is linearly dependent on the column vectors occurring to the left of it in the controllability matrix

$$\left[B \ AB \ \dots \ A^{n-1}B\right].$$

Notation for Controller Form

Given a pair (A, B), with controllability indices $\rho_1, \ldots, \rho_m$, define

$$M = \begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix} := \begin{bmatrix} B_1 \; AB_1 \; \ldots \; A^{\rho_1 - 1}B_1 \; \ldots \; B_m \; \ldots \; A^{\rho_m - 1}B_m \end{bmatrix}^{-1}$$

and the transformation matrix

$$P = \begin{bmatrix} P_1 \\ \vdots \\ P_m \end{bmatrix}, \quad P_i = \begin{bmatrix} M_{\rho_1 + \ldots + \rho_i} \\ M_{\rho_1 + \ldots + \rho_i} A \\ \vdots \\ M_{\rho_1 + \ldots + \rho_i} A^{\rho_i - 1}. \end{bmatrix}$$

Controllability Canonical Form for MIMO Systems

Suppose that (A, B) is controllable, rank $(B) = m$, and $\rho_1 + \cdots + \rho_m = n$. The transformation $\tilde{x} = Px$ constructed previously gives

$$\tilde{A} = PAP^{-1} = A_0 + B_0 F, \qquad \tilde{B} = PB = B_0 R,$$

where $F \in \mathbb{R}^{m \times n}$, and

$$A_0 = \begin{bmatrix} J_{\rho_1} & 0 & \cdots & 0 \\ 0 & J_{\rho_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{\rho_m} \end{bmatrix}, \qquad B_0 = \begin{bmatrix} e_{\rho_1} & 0 & \cdots & 0 \\ 0 & e_{\rho_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_{\rho_m} \end{bmatrix},$$

with

$$J_r = \begin{bmatrix} 0_{(r-1)\times 1} & I_{r-1} \\ 0 & 0_{1\times(r-1)} \end{bmatrix}. \qquad e_r = \begin{bmatrix} 0_{r-1} \\ 1 \end{bmatrix}.$$

Thus, the arbitrary entries of $\tilde{A}$ occur only in the last row of each block row.

Input Matrix and Brunovský Form

- $R \in \mathbb{R}^{m \times m}$ is nonsingular and upper triangular with unit diagonal:

$$R = \begin{bmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{bmatrix}.$$

  Consequently,

$$\boxed{\tilde{B} = B_0 R}$$

  and, in general, $\tilde{B} \neq B_0$.

- The transformed system is therefore

$$\dot{\tilde{x}} = (A_0 + B_0 F)\tilde{x} + B_0 R u.$$

- Introducing the feedback and input transformation

$$v = F\tilde{x} + Ru,$$

  gives the Brunovský form

$$\boxed{\dot{\tilde{x}} = A_0 \tilde{x} + B_0 v.}$$

Brunovsky Form

$$
A_0 = \left[\begin{array}{cccc|cccc|cccc}
0 & 1 & & 0 & 0 & & & 0 & 0 & & & 0 \\
 & 0 & \ddots & & & & & & & & & \\
 & & \ddots & 1 & & & & & & & & \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\hline
 & & & & 0 & 1 & & & & & & \\
 & & & & & \ddots & \ddots & & & & & \\
 & & & & 0 & \cdots & \cdots & 0 & & & & \\
\hline
 & & & & & & & & 0 & 1 & & \\
 & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & 0 & 1 \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0
\end{array}\right].
$$

$$
B_0 = \operatorname{diag}\left(e_{\rho_1}, e_{\rho_2}, \ldots, e_{\rho_m}\right), \qquad e_{\rho_i} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^\top .
$$

The Brunovsky form leads to decoupled subsystems: $y_i^{(k_i)} = u_i,\ i \in \{1, \ldots, m\}$. This form can be used to design controllers easily.

Example 3: Stabilizing an Inverted Pendulum

- Consider a torque-controlled pendulum linearized about its upright equilibrium:

$$m\ell^2\ddot{\theta} + b\dot{\theta} - mg\ell\theta = u.$$

- Let $m = \ell = 1,\ b = 0.5,\ g = 9.81,\ x = \mathrm{col}(\theta, \dot{\theta})$. Then

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 9.81 & -0.5 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u.$$

- The open-loop poles are $\lambda(A) = \{2.89, -3.39\}$, so the upright equilibrium is unstable. Two poles can be assigned arbitrarily as

$$\mathrm{rank}\begin{bmatrix} B & AB \end{bmatrix} = \mathrm{rank}\begin{bmatrix} 0 & 1 \\ 1 & -0.5 \end{bmatrix} = 2.$$

- A reasonable design target is $\zeta \approx 0.7,\ T_s \approx 2\ \mathrm{s}$, which suggests the desired poles $p_{1,2} = -2 \pm 2j$.

Assigning the Poles

- Let $u = Kx,\ K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. The closed-loop matrix is

$$A + BK = \begin{bmatrix} 0 & 1 \\ 9.81 + k_1 & -0.5 + k_2 \end{bmatrix}.$$

  The desired characteristic polynomial is
  $(s + 2 - 2j)(s + 2 + 2j) = s^2 + 4s + 8$.
- On the other hand,

$$\det\big(sI - (A + BK)\big) = \det \begin{bmatrix} s & -1 \\ -(9.81 + k_1) & s + 0.5 - k_2 \end{bmatrix}$$
$$= s^2 + (0.5 - k_2)s - (9.81 + k_1).$$

- Matching coefficients gives $0.5 - k_2 = 4,\ -(9.81 + k_1) = 8$, thus

$$\boxed{K = \begin{bmatrix} -17.81 & -3.50 \end{bmatrix}.}$$

  Thus, the feedback law $u = -17.81\,\theta - 3.50\,\dot{\theta}$ locally stabilizes the pendulum around its upright equilibrium.

Simulation Results

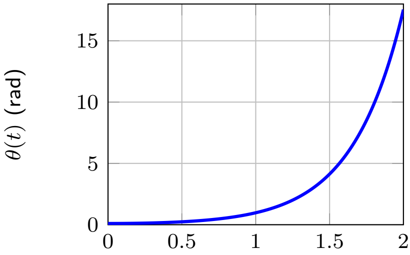
Consider the linearized inverted pendulum

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 9.81 & -0.5 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u, \qquad x(0) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}.$$

We use the controller $u = Kx, \; K = \begin{bmatrix} -17.81 & -3.50 \end{bmatrix}$.

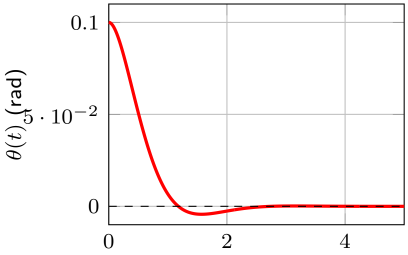
Open loop: $u = 0$

$\lambda(A) = \{2.89, -3.39\}$



Closed loop: $u = Kx$

$\lambda(A + BK) = \{-2 \pm 2j\}$



Ready to Go Beyond Pole Placement?

Eigenvalue assignment gives us stabilizing controllers. But stabilization is only the beginning.

- **Robust Control**
  ELE6214 Commande de systèmes incertains
- **Adaptive Control**
  ELE6214 Commande de systèmes incertains
- **Optimal Control**
  ELE6210 Conception de systèmes de commande
- **Nonlinear Control**
  ELE6204 Commande des systèmes non linéaires
- **Stochastic Control**
  ELE6215 Commande stochastique et filtrage
- **Model Predictive Control**

Stabilization is the first victory. Performance is the real game.

Sources and Further Reading

- State feedback and controllability decomposition: primarily on Rugh, (Callier and Desoer), and (Antsaklis and Michel).
- PBH test and stabilizability: (Antsaklis and Michel) and Hespanha.
- Controllability canonical form: The construction of the SISO controllability canonical form and its use in state-feedback design draw mainly on Rugh, (Callier and Desoer), and Hespanha.
- Eigenvalue assignment: The pole-placement theorem and the relation among controllability, arbitrary eigenvalue assignment, and closed-loop stabilization follow standard results in Rugh and (Antsaklis and Michel).
- MIMO controller and Brunovský forms: The treatment using controllability indices, input transformations, and chains of integrators follows Rugh, particularly Sec. 13.9.

The inverted-pendulum design and simulations were prepared by the instructor and ChatGPT. Selected proofs and constructions were expanded and reorganized for the objectives of this course.