ELE6202E - Multivariable Systems

Lecture 7: State Observer and Output Feedback

Fall 2026

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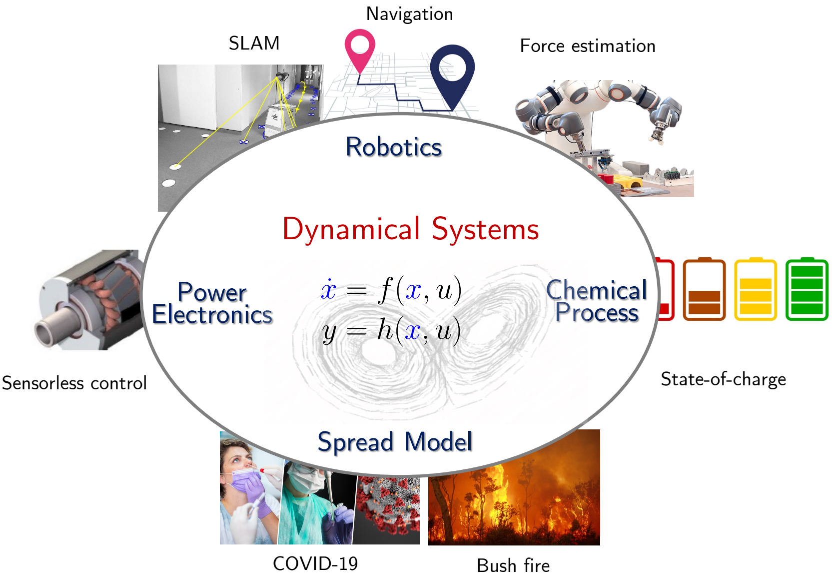
https://bwyi.github.io

Contents

1. Motivation

Internal/Hidden Variables



Motivation

In many applications, it is not practical to measure all of the states directly and we can measure only a small number of outputs (corresponding to the sensors that are available):

$$y = C(t)x + D(t)u$$

- $y \in \mathbb{R}^p$ and $p < n$
- We could try to find a static output feedback

  $$u = Ky + r,$$

  but it is in general difficult to determine. (Note $\dot{x} = Ax + BKy = (A + BKC)x$, being difficult to assign eigenvalues.)
- Instead we try to reconstruct $x(t)$ approximately, using $y(s)$, $u(s)$ and the model with $0 \le s \le t$.

Estimation: Practical Applications

1. Output feedback control
2. Monitoring
3. Fault detection
4. Navigation and localization (Robotics & aircraft)
5. ...

Two Categories of Approaches

- Optimization-based
  For the DT system

$$x_{k+1} = A_k x_k + B_k u_k,\ y_k = C_k x_k,$$

  define the vector with historical information

$$\mathbf{x} := [x_1^\top, x_2^\top, \ldots, x_k^\top]^\top$$

  and solve

$$\begin{aligned} &\arg\min_{\hat{\mathbf{x}} \in \mathbb{R}^{nk}} \sum_{i=1}^{k} \|y_i - C_i \hat{x}_i\|^2 \\ &\text{s.t. } \hat{x}_{k+1} = A_k \hat{x}_k + B_k u_k \end{aligned}$$

- Observer/Filtering
  Recursive design (see the next slide)

Problem Formulation of State Estimation

Given a linear system model, finding a dynamical linear system

$$\dot{\hat{x}} = F\hat{x} + Gu + Hy$$

s.t.

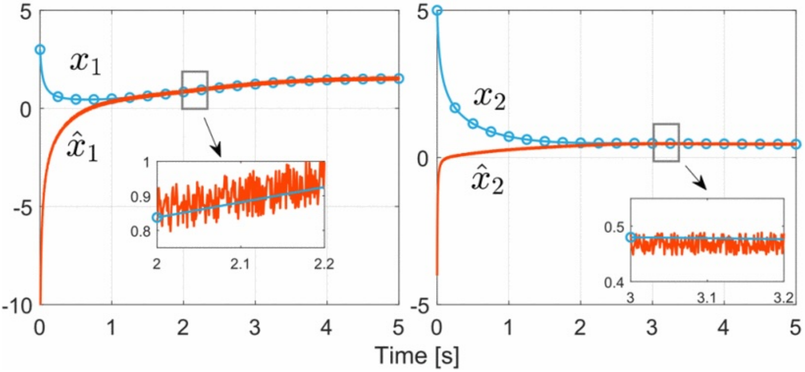
$$\lim_{t \to +\infty} |x(t) - \hat{x}(t)| = 0.$$

We call the above dynamics as a Luenberger observer.

Measurements

Remark: We do not model the presence of noise in $y = Cx + Du + v$ and $\dot{x} = Ax + Bu + w$.

Asymptotically Convergent Estimates



2. Brief Review of Observability

Review: Observability

Consider the LTV system

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u.$$

- Observability refers to the ability to determine $x(t_0)$ from $u(t)$ and $y(t)$ on $[t_0, t_1]$.
- The observability gramian for the pair $(A(t), C(t))$ on $[t_0, t_f]$ is

$$W_O(t_0, t_f) = \int_{t_0}^{t_f} \Phi^\top(s, t_0) C^\top(s) C(s) \Phi(s, t_0) ds.$$

- System is observable on $[t_0, t_f] \iff W_O(t_0, t_f)$ is invertible.
- (A, C) is observable $\iff (A^\top, C^\top)$ is controllable.

- For LTI systems, observable $\iff$ rank $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$
- PBH Test: (A, C) is observable $\iff$ there is no eigenvector of A in the null space of C, i.e.

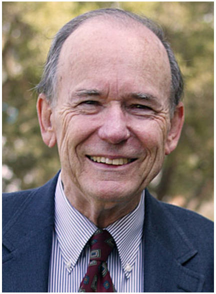
$$\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C}.$$

- If A is Hurwitz, (A, C) is observable $\iff$ the ALE $A^\top W + WA = -C^\top C$ has a unique p.d. solution.

  [DT: A is Schur stable, the ALE becomes $A^\top WA - W = -C^\top C$.]
- Observable Canonical Form: I will show you later on ...

3. State Observers

Luenberger Observer



David Luenberger (Sept 1937 – May 2026)

David Gilbert Luenberger was a mathematical scientist on mathematical optimization. He was a professor at Stanford University. His PhD thesis, titled ”Determining the State of a Linear System with Observers of Low Dynamic Order,” introduced new methods for construction of state observers. The celebrated Luenberger observer is named after him.

Observers

Consider the given system

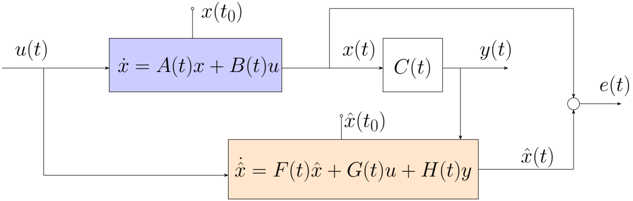
$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u.$$

We want to design an observer in the form of

$$\dot{\hat{x}} = F(t)\hat{x} + G(t)u + H(t)y.$$

Two requirements:

1. [Invariance] If $\hat{x}_0 = x_0$, then $\hat{x}(t) = x(t)$ for all $t \geq t_0$.
2. [Convergence] $\hat{x}(t) \to x(t)$ as $t \to \infty$.

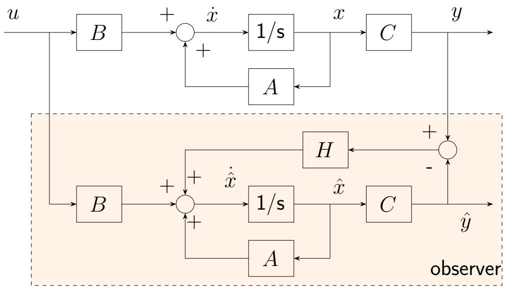


- To guarantee $x_0 = \hat{x}_0 \implies x(t) \equiv \hat{x}(t)$, we have $F(t)x + G(t)u + H(t)C(t)x = A(t)x + B(t)u$, thus

$$\begin{aligned} G(t) &= B(t) \\ F(t) &= A(t) - H(t)C(t). \end{aligned} \tag{1}$$

- The observer can be reformulated as

$$\begin{aligned} &\dot{\hat{x}} = A(t)\hat{x} + B(t)u + H(t)[y - \hat{y}], \quad \hat{x}(t_0) = \hat{x}_0 \\ &\hat{y} = C(t)\hat{x}. \end{aligned} \tag{2}$$



- Estimation error dynamics $e := \hat{x} - x$:

$$\dot{e} = [A(t) - H(t)C(t)]e, \quad e(t_0) = \hat{x}_0 - x_0.$$

The second condition is satisfied if $H(t)$ is chosen s.t. $(A - HC)$ is uniformly exponentially stable.

Dual problem of stabilization.

Existence of an Observer

Theorem 1 *(Rugh, page 267, Existence Result)* Suppose for the LTV system, $\exists\ \delta, \alpha_1, \alpha_2 > 0$ s.t.

$$\alpha_1 I \preceq \Phi^\top(t-\delta,t) W_O(t-\delta,t)\Phi(t-\delta,t) \preceq \alpha_2 I, \quad \forall t. \tag{3}$$

Then, given $\gamma > 0$ the observer gain

$$H(t) = [\Phi^\top(t-\delta,t)W_\alpha(t-\delta,t)\Phi(t-\delta,t)]^{-1}C^\top(t).$$

with

$$W_\alpha(t_1,t_2) = \int_{t_1}^{t_2} 2e^{4\gamma(s-t_2)}\Phi^\top(s,t_1)C^\top(s)C(s)\Phi(s,t_1)ds$$

makes $[A(t) - H(t)C(t)]$ UES.

4. Observers for LTI Systems

Luenberger Observer for LTI Systems

- Consider the LTI system

$$\begin{aligned}\dot{x} &= Ax + Bu,\ x(0) = x_0\\ y &= Cx.\end{aligned}$$

A Luenberger observer is in the form

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + H[y - \hat{y}],\ \hat{x}(0) = \hat{x}_0\\ \hat{y} &= C\hat{x}\end{aligned}$$

- As long as H is selected s.t. $(A - HC)$ is Hurwitz, the estimation error converges to zero exponentially.
- Observability $\implies$ $\lambda_i(A - HC)$ is arbitrarily assignable.

However, observability is unnecessary for $\lambda_i(A - HC) \in \mathbb{C}_-$; instead we only need detectability.

Observable Canonical Form (SISO)

For the single-input-single-output systems, (A, C) is in observable canonical form (if not, a similarity transform can help out):

$$A = \begin{bmatrix} -\alpha_{n-1} & 1 & 0 & \cdots \\ \vdots & 0 & 1 & \cdots \\ -\alpha_1 & \vdots & \ddots & 1 \\ -\alpha_0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_{n-1} \\ \vdots \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}, \qquad D = d$$

$$\det(sI - A) = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

Observability Decomposition

Observability is the dual of controllability. The unobservable subspace

$$\bar{\mathcal{O}} = \{x_0 \in \mathbb{R}^n : C\Phi(t,t_0)x_0 = 0, \forall t\}$$

is equivalent to

$$\bar{\mathcal{O}} = N(C) \cap N(CA) \cap \ldots \cap N(CA^{n-1}),$$

which is the set of states that are indistinguishable from the origin.

If $\{0\} \subsetneq \bar{\mathcal{O}}$, we can choose V_2 s.t.

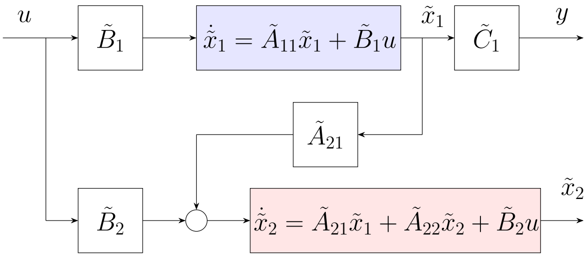
$$\mathbb{R}^n = V_2 \oplus \bar{\mathcal{O}}.$$

Similar to controllable decomposition, we can represent in the new coordinate s.t.

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},\ \tilde{C} = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix}.$$

The LTI system can thus be written as

$$
\begin{aligned}
\dot{\tilde{x}}_1 &= \tilde{A}_{11}\tilde{x}_1 + \tilde{B}_1 u \\
\dot{\tilde{x}}_2 &= \tilde{A}_{21}\tilde{x}_1 + \tilde{A}_{22}\tilde{x}_2 + \tilde{B}_2 u \\
y &= \tilde{C}_1\tilde{x}_1.
\end{aligned}
\tag{4}
$$



Now $(\tilde{A}_{11}, \tilde{C}_1)$ is completely observable and the eigenvalues of $\tilde{A}_{22}$ are the unobservable modes, i.e.

$$\lambda(A) = \lambda(\tilde{A}) = \underbrace{\lambda(\tilde{A}_{11})}_{\text{observable modes}} \cup \underbrace{\lambda(\tilde{A}_{22})}_{\text{unobservable modes}}$$

Definition 2 (Detectability) The pair (A, C) is detectable if all unobservable modes are stable, i.e.

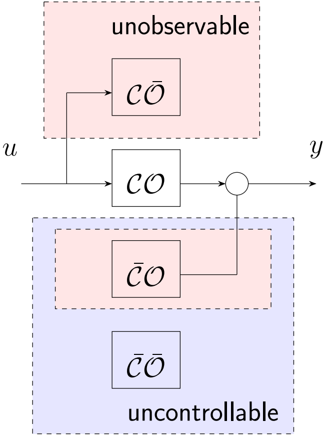
$$\text{rank} \begin{bmatrix} sI - A \\ C \end{bmatrix} = n, \ \forall s \in \mathbb{C}_+.$$

Kalman Decomposition

The Kalman decomposition combines the controllable/uncontrollable and observable/unobservable decompositions.

Every state-space model can be transformed, by equivalence transformation, into a canonical form that splits the states into

- Controllable and observable states
- Controllable and unobservable states
- Uncontrollable but observable states
- Uncontrollable and unobservable states



Kalman Decomposition (cont'd)

The representation of A, B, C via transformation is

$$\tilde{A} = \left[\begin{array}{c|c|c|c} \tilde{A}_{11} & 0 & \tilde{A}_{13} & 0 \\ \hline \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ \hline 0 & 0 & \tilde{A}_{33} & 0 \\ \hline 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} \end{array}\right]$$

$$\tilde{B} = \left[\begin{array}{c} \tilde{B}_1 \\ \hline \tilde{B}_2 \\ \hline 0 \\ \hline 0 \end{array}\right]$$

$$\tilde{C} = \left[\begin{array}{c|c|c|c} \tilde{C}_1 & 0 & \tilde{C}_3 & 0 \end{array}\right]$$

Rudolf Kalman (1930-2016)

5. Output Feedback

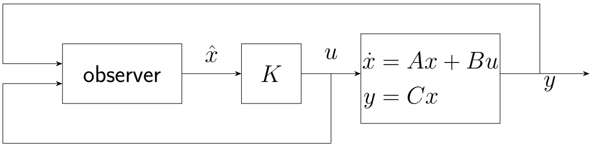
Observer-based Output Feedback

Given system dynamics:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

- State feedback control: arbitrary eigenvalue assignment if the system is controllable;
- Observer design: arbitrary observer eigenvalue assignment for state estimation if the system is observable;
- When full states are not available, what may happen if we combine both?

$$u = -K\hat{x} + r.$$



Closed-Loop Dynamics

- Full closed-loop system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ \dot{\hat{x}} &= A\hat{x} + Bu + H(y - C\hat{x}) \\ u &= -K\hat{x} + r \end{aligned}$$

- Using the similarity transform $\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -I_n & I_n \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$ gives

$$\frac{d}{dt}\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A - BK & -BK \\ 0 & A - HC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

Note that the modes of the composite system is

$$\lambda(A - BK) \cup \lambda(A - HC).$$

- We can design K and H separately based on the previously discussed tools.
- (A, B) stabilizable and (A, C) detectable.

We conclude the Separation Principle:

> The controller and observer can be designed separately.

6. Reduced-Order Observer

Reduced-Order Observer

Partial states are measured with the partition:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$
$$y = \begin{bmatrix} I_p & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \in \mathbb{R}^p. \tag{5}$$

Questions:

- We do not need to design an n-dim observer to fully estimate x.
- Can we design an observer with an $(n-p)$-dimensional dynamics to estimate x_2 directly?

Problem Formulation

We want to design a reduced-order observer

$$\begin{aligned}\dot{\hat{z}} &= F\hat{z} + Gu + Hy \\ \hat{x}_2 &= \hat{z} + Ny\end{aligned}$$

with $\hat{z} \in \mathbb{R}^{n-p}$ and $N \in \mathbb{R}^{(n-p)\times p}$ full rank, satisfying

$$\lim_{t\to+\infty} |\hat{x}_2(t) - x_2(t)| = 0.$$

The core idea for reduced-order observers: using a different coordinate z rather than designing for x_2 directly.

Let us consider the transformed coordinate:

$$x = \begin{bmatrix} I_p & 0 \\ N & I_{n-p} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}.$$

($\hat{z}$ in the observer can be viewed as the estimate of z.)

1. [Invariance] If $\hat{x}_2(t_0) = x_2(t_0)$, then $\hat{x}_2(t) = x_2(t)$ for all $t \geq t_0$.
2. [Convergence] $\hat{x}_2(t) \to x_2(t)$ as $t \to \infty$.

Define the (partial) estimation error $\tilde{x}_2 = \hat{x}_2 - x_2$, s.t.

$$\begin{aligned}
\dot{\tilde{x}}_2 &= \dot{\hat{z}} + N\dot{y} - A_{21}x_1 - A_{22}x_2 - B_2 u \\
&= F\hat{x}_2 + (NA_{12} - A_{22})x_2 + (-FN + H + NA_{11} - A_{21})x_1 + \\
&\quad + (G - B_2 + NB_1)u.
\end{aligned}$$

The reduced-order observer is thus given by

$$\dot{\hat{z}} = F\hat{z} + Gu + Hy$$
$$\hat{x}_2 = \hat{z} + Ny$$

with

$$F = A_{22} - NA_{12}$$
$$G = B_2 - NB_1$$
$$H = A_{21} + FN - NA_{11}.$$

The error dynamics is

$$\dot{\tilde{x}}_2 = (A_{22} - NA_{12})\tilde{x}_2$$

We need to find an N to stabilize the above error dynamics!

Proposition 1 If (A, C) is completely observable, then (A_{22}, A_{12}) is c.o.

Proof.

Recall the observability rank condition:

$$(A, C) \text{ c.o.} \quad \iff \quad \nexists \lambda \in \mathbb{C},\ v \neq 0 : \ Av = \lambda v \text{ and } Cv = 0.$$

Suppose (A_{22}, A_{12}) is not c.o., then $\exists\ \lambda, p \neq 0 : A_{22}p = \lambda p$ and $A_{12}p = 0$. By taking $v = \begin{bmatrix} 0 \\ p \end{bmatrix} \neq 0$, we have

$$Av = \begin{bmatrix} A_{12}p \\ A_{22}p \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ p \end{bmatrix} = \lambda v \text{ and } \underbrace{\begin{bmatrix} I_p & 0 \end{bmatrix}}_{C} v = 0.$$

Hence, (A, C) is not c.o, which is a contradiction. ∎

Functional Observers

In some cases, only part of or a linear combination of states is of interests: $w = Tx$ with $T \in \mathbb{R}^{\rho \times n}$ $(\rho \leq n)$. We want to design a functional observer

$$\dot{z} = Fz + Gu + Hy$$
$$\hat{w} = T_1(y - Du) + T_2 z$$

to estimate w directly.

- Reduced-order observer is a special case of functional observers.
- Details are given in Antsaklis's textbook (page 355).

Discrete-Time Observers

- Analogous to the CT case: full state feedback is usually not available.
- Consider the DT model:

  $$x_{k+1} = Ax_k + Bu_k,\ y_k = Cx_k.$$

  The DT Luenberger observer can be designed as

  $$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + H(y_k - C\hat{x}_k)$$

- The error variable: $e_k = \hat{x}_k - x_k$, whose dynamics is

  $$e_{k+1} = [A - HC]e_k$$

- Overall dynamics:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A - HC \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_k.$$

- The error dynamics can be arbitrarily assigned if the system is observable.
- Observable $\iff$ the observability matrix $\mathcal{O}$ is full rank:

$$\mathbb{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- Observability gramian:

$$W_O[k_0, k_j] = \sum_{j=k_0}^{k_j - 1} \Phi^\top(j, k_0) C^\top(j) C(j) \Phi(j, k_0).$$

7. Kalman Filter ($\star$)

Kalman Filter $\star$

We have shown how to get x_0 in the ideal case. In practice, there are noises that corrupt the measurement $y(t)$. Define the operator

$$\begin{aligned} \mathcal{L}_o : \mathbb{R}^n &\to PC^m[t_0, t_1] \\ x_0 &\mapsto C(\cdot)\Phi(\cdot, t_0)x_0 \end{aligned}$$

With noise, $y(\cdot)$ may not be in $\mathrm{Im}\,(\mathcal{L}_o)$.

For simplicity, we consider the LTV system

$$\dot{x} = A(t)x, \quad y = C(t)x + n(t),$$

with $n(t)$ noise process. Now, $y(\cdot) \notin \mathrm{Im}\,(\mathcal{L}_o)$, we want to minimize $\|y - \mathcal{L}_o x_0\|^2$.

In linear space, if $W_o[t_0, t_1]$ is full, then

$$\hat{x}_0 = (\mathcal{L}_o^* \mathcal{L}_o)^{-1} \mathcal{L}_o^* y$$

minimizes $\|y(\cdot) - \mathcal{L}_o x_0\|^2$. On $PC^m[t_0, t_1]$,

$$\begin{aligned} \mathcal{L}_o^* \mathcal{L}_o &= W_o[t_0, t_1], \\ \mathcal{L}_o^* y &= \int_{t_0}^{t_1} \Phi^\top(s, t_0) C^\top(s) y(s) ds. \end{aligned}$$

Then, the optimal estimate of IC is

$$\hat{x}(t_0) = W_o^{-1}[t_0, t] \mathcal{L}_o^* y(\cdot)|_{[t_0,t]} = W_o^{-1}[t_0, t] \int_{t_0}^{t} \Phi^\top(\tau, t_0) C^\top(\tau) y(\tau) d\tau,$$

$$\hat{x}(t) = \Phi(t, t_0) \hat{x}(t_0) = \Phi(t, t_0) W_o^{-1}[t_0, t] \int_{t_0}^{t} \Phi^\top(\tau, t_0) C^\top(\tau) y(\tau) d\tau.$$

To obtain the filter dynamics, derive the dynamics of $\hat{x}(t)$:[1]

$$\begin{aligned}
\dot{\hat{x}} =& A_t\hat{x}_t + \Phi(t,t_0)\dot{W}_o^{-1}[t_0,t]\int_{t_0}^{t}\Phi^\top(\tau,t_0)C_\tau^\top y_\tau d\tau + \Phi(t,t_0)W_o^{-1}\Phi^\top(t,t_0)C_t^\top y_t \\
=& A_t\hat{x}_t - \Phi(t,t_0)W_o^{-1}\Phi^\top(t,t_0)C_t^\top C_t\Phi(t,t_0)W_o^{-1}\int_{t_0}^{t}\Phi^\top(\tau,t_0)C_\tau^\top y_\tau d\tau \\
& + \Phi(t,t_0)W_o^{-1}\Phi^\top(t,t_0)C_t^\top y_t. \\
=& A_t\hat{x}_t - \Phi(t,t_0)W_o^{-1}\Phi^\top(t,t_0)C_t^\top C_t\hat{x}_t + \Phi(t,t_0)W_o^{-1}\Phi^\top(t,t_0)C_t^\top y_t
\end{aligned}$$

with $A_t := A(t)$ and $W_o := W_o[t_0,t]$.

[1]To obtain $\dot{W}_o^{-1}[t_0,t]$, we use the relations $\dot{W}_o^{-1} = -W_o^{-1}\dot{W}_oW_o^{-1}$ and

$$\frac{d}{dt}W_o[t_0,t] = \Phi^\top(t,t_0)C^\top(t)C(t)\Phi(t,t_0).$$

Kalman Filter

It yields

$$\dot{\hat{x}} = A(t)\hat{x} + K(t)[y - C(t)\hat{x}], \tag{6}$$

with the Kalman Gain

$$K(t) = \Phi(t, t_0) W_o^{-1} \Phi^\top(t, t_0) C^\top(t).$$

Please compare the form (6) with Luenberger observer!

Computing Kalman Gain

The gain $K(t) = \Phi(t, t_0) W_o^{-1} \Phi^\top(t, t_0) C^\top(t)$ can be written as

$$\begin{aligned} K(t) &= P(t) C^\top(t) \\ P &:= \Phi(t, t_0) W_o^{-1} \Phi^\top(t, t_0) \end{aligned}$$

The matrix P satisfies the ODE:

$$\begin{aligned} \dot{P} &= A(t)P + PA^\top(t) \\ &\quad + \Phi(t, t_0) \underbrace{\left[-W_o^{-1} \Phi^\top(t, t_0) C^\top(t) C(t) \Phi(t, t_0) W_o^{-1}\right]}_{\dot{W}_o^{-1}} \Phi^\top(t, t_0) \\ &= A(t)P + PA^\top(t) - PC^\top C(t) P \end{aligned}$$

Computing Kalman Gain (cont'd)

The gain $K(t)$ now becomes

$$K(t) = P(t)C^\top(t)$$
$$\dot{P} = A(t)P + PA^\top(t) - PC^\top C(t)P, \quad P(0) = W_o^{-1}(t_0, t_0) = \infty?$$

The IC $P(0)$ is not well defined!

Define the inverse $\Sigma(t) := P^{-1}(t)$, then

$$\dot{\Sigma} = -\Sigma A(t) - A^\top(t)\Sigma + C^\top(t)C(t), \qquad \Sigma(t_0) = 0_{n\times n}.$$

- Dynamics of P: Differential Riccati Equation
- Dynamics of Σ: Differential Lyapunov Equation.

Kalman-Bucy Filter/Observer

In conclusion, we obtain the following optimal observer in the presence of noise:

$$\boxed{\begin{aligned}\dot{\hat{x}} &= A(t)\hat{x} + K[y - C(t)\hat{x}]\\ K(t) &= \Sigma^{-1}(t)C^\top(t)\\ \dot{\Sigma} &= -\Sigma A(t) - A^\top(t)\Sigma + C^\top(t)C(t), \qquad \Sigma(t_0) = 0.\end{aligned}}$$

This is (a simplified version of) the famous Kalman-Bucy filter.

Riccati Observer

Consider the deterministic LTV system

$$\dot{x} = A(t)x + B(t)u, \qquad y = C(t)x + D(t)u,$$

where $u(t)$ is known.

Riccati observer:

$$\begin{aligned} \dot{\hat{x}} &= A(t)\hat{x} + B(t)u + H(t)\big[y - C(t)\hat{x} - D(t)u\big], \\ H(t) &= P(t)C^\top(t)R^{-1}, \\ \dot{P} &= A(t)P + PA^\top(t) + Q - PC^\top(t)R^{-1}C(t)P, \\ P(t_0) &= P_0 \succ 0, \end{aligned}$$

where $Q = Q^\top \succ 0$ and $R = R^\top \succ 0$ are constant design matrices.

- Under boundedness and UCO assumptions, it provides exponentially convergent estimates.
- No stochastic model is required!

Example 1: State Estimation for a DC Motor

- Consider the simplified rotational dynamics

$$J\ddot{\theta} + b\dot{\theta} = u,$$

where θ is the angular position, $\dot{\theta}$ is the angular velocity, and u is the applied torque. Let $J = 1$ and $b = 0.5$. With $x = \mathrm{col}(\theta, \dot{\theta})$, the state-space model is

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u, \qquad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x.$$

- Only the angular position is measured, with the angular velocity to estimate.

- The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and hence (A, C) is observable.

Estimator 1: Luenberger Observer

- Consider the observer
$$\dot{\hat{x}} = A\hat{x} + Bu + H_L(y - C\hat{x}).$$
- Choose the observer poles as $\lambda(A - H_L C) = \{-3, -4\}$. Let $H_L = \text{col}(h_1, h_2)$. Then
$$A - H_L C = \begin{bmatrix} -h_1 & 1 \\ -h_2 & -0.5 \end{bmatrix},$$
and $\det\big(sI - (A - H_L C)\big) = s^2 + (h_1 + 0.5)s + (0.5h_1 + h_2)$. Matching it with $(s+3)(s+4) = s^2 + 7s + 12$ gives
$$H_L = \begin{bmatrix} 6.5 & 8.75 \end{bmatrix}^\top.$$
- Therefore, the Luenberger observer is given by
$$\boxed{\begin{aligned} \dot{\hat{\theta}} &= \hat{\omega} + 6.5(y - \hat{\theta}), \\ \dot{\hat{\omega}} &= -0.5\hat{\omega} + u + 8.75(y - \hat{\theta}). \end{aligned}}$$

Estimator 2: Reduced-Order Observer

- Since $y = \theta = x_1$ is measured, we only need to estimate $x_2 = \omega$.
- The system partition is

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u = x_2,$$
$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2 u = -0.5x_2 + u,$$

  where $A_{11} = 0,\ A_{12} = 1,\ A_{21} = 0,\ A_{22} = -0.5,\ B_1 = 0,\ B_2 = 1.$
- A reduced-order observer is

$$\dot{\hat{z}} = F\hat{z} + Gu + H_r y, \qquad \hat{\omega} = \hat{z} + Ny,$$

  with $F = A_{22} - NA_{12}$. Choose the observer pole at -4. Then $-0.5 - N = -4 \implies N = 3.5$. Thus, $F = -4,\ G = B_2 - NB_1 = 1,$ and $H_r = A_{21} + FN - NA_{11} = -14.$
- A reduced-order observer is designed as

$$\boxed{\begin{aligned} \dot{\hat{z}} &= -4\hat{z} + u - 14y, \\ \hat{\omega} &= \hat{z} + 3.5y. \end{aligned}}$$

Estimator 3: Kalman–Bucy Filter

- Choose the design matrices $Q = \text{diag}(1,4), R = 0.24, P(0) = I$, and solve the differential Riccati equation

$$\dot{P} = AP + PA^\top + Q - PC^\top R^{-1} CP.$$

- Define the time-varying observer gain $H_K(t) = P(t)C^\top R^{-1}$. Since $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $R^{-1} = 4$,

$$H_K(t) = 4 \begin{bmatrix} p_{11}(t) \\ p_{12}(t) \end{bmatrix} =: \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}.$$

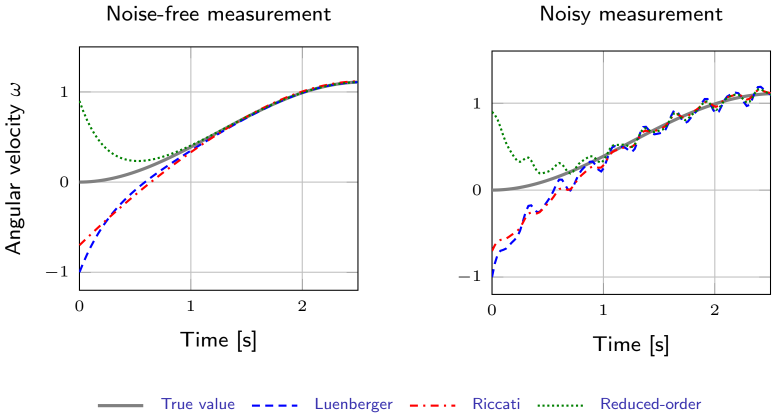
- Therefore, the Riccati observer is given by

$$\boxed{\begin{aligned} \dot{\hat{\theta}} &= \hat{\omega} + h_1(t)(y - \hat{\theta}), \\ \dot{\hat{\omega}} &= -0.5\hat{\omega} + u + h_2(t)(y - \hat{\theta}). \end{aligned}}$$

- The Riccati solution converges to

$$P_\infty \approx \begin{bmatrix} 0.7587 & 0.6514 \\ 0.6514 & 2.3027 \end{bmatrix}, \qquad H_K(t) \longrightarrow H_{K,\infty} \approx \begin{bmatrix} 3.035 \\ 2.606 \end{bmatrix}.$$

Simulation Comparison of Three Estimators



All three estimators converge in the noise-free case. The high-gain Luenberger observer is more sensitive to measurement noise.

Sources and Further Reading

- Observability, detectability, and decompositions: The review draws primarily on Rugh.
- Luenberger observers: It follows Rugh and Luenberger's original observer framework.[2]
- Observer-based output feedback: Rugh and (Antsaklis and Michel).
- Reduced-order and functional observers: (Antsaklis and Michel).
- Riccati observers and Kalman–Bucy filtering: The least-squares interpretation, differential Riccati equation, deterministic Riccati observer, and simplified Kalman–Bucy formulation draw on classical continuous-time filtering theory.[3]

Selected derivations, block diagrams, and simulation comparisons were expanded and reorganized by the instructor.

[2] D. G. Luenberger, "Observing the State of a Linear System," *IEEE Transactions on Military Electronics*, 1964.

[3] R. E. Kalman and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory," *Journal of Basic Engineering*, 1961.