ELE6202E - Multivariable Systems

Lecture 8: Elements of Realization Theory

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POLYTECHNIQUE MONTRÉAL

Contents

1. Realization Concepts

I/O Representation

Given a state-space model

$$\dot{x} = A_t x + B_t u, \quad y = C_t x + D_t u,$$

it is straightforward to obtain the input-output (I/O) representation.

- LTV: with the STM Φ, it includes the zero-input response and the zero-state response.

$$y_t = \underbrace{C_t \Phi(t, t_0) x(t_0)}_{\text{zero-input resp.}} + \underbrace{\int_{t_0}^{t} [C_t \Phi(t, \tau) B(\tau) + D_t \delta(t - \tau)] u(\tau) d\tau}_{\text{zero-state response}}.$$

with the impulse response

$$H(t, \tau) = \begin{cases} C_t \Phi(t, \tau) B(\tau) + D_t \delta(t - \tau), & t \geq \tau \\ 0, & t < \tau. \end{cases}$$

- LTI: The impulse response becomes

$$H(t,\tau) = \begin{cases} Ce^{A(t-\tau)}B + D\delta(t-\tau), & t \geq \tau \\ 0, & t < \tau. \end{cases}$$

It can also be represented by the transfer function:

$$G(s) = \mathcal{L}[H(t)] = C(sI - A)^{-1}B + D.$$

Realizability is about the opposite direction:

I/O representation → State-space model

Realization

Definition 1 Given the impulse response $(t, \tau) \mapsto H(t, \tau)$, a realization of H is any (A_t, B_t, C_t, D_t) such that the I/O response relation above is satisfied.

Definition 2 (Realization for LTI Systems) For LTI systems, assume $\hat{H}(s)$ is a given proper transfer function (TF) matrix. We say that an LTI system representation (A, B, C, D) is a realization if the TF is $\hat{H}(s)$, i.e.

$$\hat{H}(s) = C(sI - A)^{-1}B + D.$$

A realization allows us to build a system (e.g. computer program, a circuit) that achieves a certain I/O behavior.

Realization: Non-Uniqueness

- First, we need conditions on H under which a realization exists.
- If it exists, then we have many realizations:
  1. We can change coordinates by $x = Pz$ into

     $$\dot{z} = (\underbrace{P^{-1}AP - P^{-1}\dot{P}}_{\tilde{A}})z + \underbrace{P^{-1}B}_{\tilde{B}}\,u.$$

     Recall $\Phi_z(t,\tau) = P^{-1}(t)\Phi(t,\tau)P(\tau)$, then $\tilde{C}(t)\Phi_z(t,\tau)\tilde{B}(\tau) = C(t)\Phi(t,\tau)B(\tau)$, $\forall t,\tau$. This does not change H, or controllability/observability.
  2. We can always get a realization with $A \equiv 0$ by using a time-varying change of variable:

     $$x(t) = \Phi(t,t_0)z(t) \implies \dot{x} = Ax + \Phi(t,t_0)\dot{z} = Ax + Bu,$$

     Then, we have

     $$\dot{z} = \Phi(t,t_0)^{-1}B(t)u(t).$$

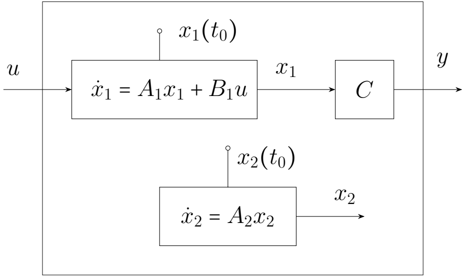
3. The state-space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u, \; y = \begin{bmatrix} C_1 & 0 \end{bmatrix} x$$

has the TF

$$H(s) = C_1(sI - A_1)^{-1}B_1,$$

no matter the choice of A_2. (Different-dimensional realization.)



Definition 3 We call the dimension of any representation (A_t, B_t, C_t, D_t) the dimension n of its state space. We say that a realization of $\hat{H}(s)$ is minimal if its dimension is minimal among all realizations of $\hat{H}(s)$.

- The system has the transfer matrix/function $G(s)$ defined as

$$G(s) = C(sI - A)^{-1}B + D, \tag{1}$$

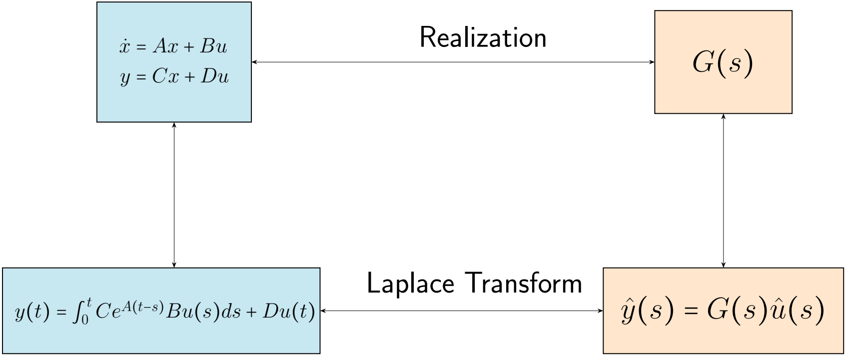
which is a matrix whose elements consist of real-rational and proper functions in s.

- The fundamental relation between the state-space model and frequency domain representation is studied in realization theory.

$$G(s) = C_G(sI - A_G)^{-1}B_G + D_G,$$

A feasible realization (A_G, B_G, C_G, D_G) may not be minimal; a minimal realization is controllable and observable.

- View the system as a device that processes signals from input $u(\cdot)$ to output $y(\cdot)$.



$\dot{x} = Ax + Bu$
$y = Cx + Du$
Realization
$G(s)$
$y(t) = \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t)$
Laplace Transform
$\hat{y}(s) = G(s)\hat{u}(s)$

2. Existence/Realizability and Minimal Realization

Realizability

Definition 4 (Realizability) If a realization exists, then the system is realizable.

We consider the following case:

- Since $D(t)$ plays an unessential role in realization, we consider the case with $D(t) = 0$.
- Regularity: A_t, B_t and C_t are assumed piecewise continuous.
- Impulse responses $H(t, \tau)$ need only be defined for $t \geq \tau$. However, its value for $t < \tau$ might not be completely determined, bringing mathematical issues. (Rugh, Ex. 10.7)

  Hence, we define $H(t, \tau)$ for all t, τ, and call it weighting pattern.

Theorem 5 *(Existence/Realizability)* A continuous weighting pattern $H(t,\tau)$ is realizable by a continuous LTV system (A_t, B_t, C_t) if and only if $\exists$ continuous functions $M(t)$ and $N(\tau)$ s.t.

$$H(t,\tau) = M(t)N(\tau), \quad \forall t,\tau. \tag{2}$$

Proof.

$(\Rightarrow)$ $H(t,\tau) = C(t)\Phi(t,\tau)B(\tau)$, and thus

$$H(t,\tau) = \underbrace{C(t)\Phi(t,0)}_{:=M(t)}\underbrace{\Phi(0,\tau)B(\tau)}_{:=N(\tau)}.$$

$(\Leftarrow)$ We take $A(t) \equiv 0$, so $\Phi_A = I$. A feasible realization is given by

$$\dot{x} = N(t)u$$
$$y = M(t)x.$$

■

- The above provides the basic realizability criterion for weighting patterns, often not very useful. (Difficult to check factorization)
- The obtained realization may be unsatisfactory. Consider

$$H(t,\tau) = e^{-(t-\tau)}.$$

  The above construction yields

$$\dot{x} = e^{t}u,\ y = e^{-t}x,$$

  which is an LTV system. In fact, it admits an LTI realization

$$\dot{x} = -x + u, \quad y = x.$$

Minimal Realization

Theorem 6 *(Minimal Realization)* Let (A_t, B_t, C_t) be a realization of $H(t, \tau)$. Then, it is minimal iff $\exists t_0,\ t_1 > t_0$ s.t. it is controllable and observable on $[t_0, t_1]$.

Proof.

($\Leftarrow$) Via contradiction. Assume the realization is controllable and observable on $[t_0, t_1]$, but not minimal. Then, $\exists$ a lower-dimensional realization

$$\dot{z} = F(t)z + R(t)u,\ y = S(t)z$$

with dim $F = v <$ dim $A = n$. For any fixed t_0, we get

$$H(t,\tau) = \underbrace{C(t)\Phi_A(t,t_0)}_{\Psi(t)\in\mathbb{R}^{p\times n}}\underbrace{\Phi_A(t_0,\tau)B(\tau)}_{\Gamma(\tau)\in\mathbb{R}^{n\times m}} = \underbrace{S(t)\Phi_F(t,t_0)}_{\tilde{\Psi}(t)\in\mathbb{R}^{p\times v}}\underbrace{\Phi_F(t_0,\tau)R(\tau)}_{\tilde{\Gamma}(\tau)\in\mathbb{R}^{v\times m}}.$$

This implies

$$\Psi^\top(t)\Psi(t)\Gamma(\tau)\Gamma^\top(\tau) = \Psi^\top(t)\tilde{\Psi}(t)\tilde{\Gamma}(\tau)\Gamma^\top(\tau).$$

Integrating w.r.t. t and τ yields

$$W_o(t_0,t_1)W_c(t_0,t_1) = \int_{t_0}^{t_1}\Psi^\top(t)\tilde{\Psi}(t)dt\int_{t_0}^{t_1}\tilde{\Gamma}(\tau)\Gamma^\top(\tau)d\tau$$

Since $v < n$, W_o and W_c cannot both be invertible. This leads to contradiction.

($\Rightarrow$) Assume (A_t, B_t, C_t) is minimal. By change of variable as discussed above, i.e. $x = \Phi(t,t_0)z$, we have w.l.g.

$$\tilde{A} := 0, \quad \tilde{B} := \Phi(t,t_0)^{-1}B(t),\ \tilde{\Phi}(t,t_0) = I.$$

In the new coordinate, the controllability gramian is

$$W_c(t_1,t_2) = \int_{t_1}^{t_2}\tilde{B}(\tau)\tilde{B}^\top(\tau)d\tau.$$

Prove via contradiction. Suppose $W_c(-k,k)$ is singular (uncontrollable), then $\exists x_k$ s.t. $\|x_k\| = 1$ and

$$x_k^\top W_c x_k = \int_{-k}^{k} \|\tilde{B}^\top(t) x_k\|^2 dt = 0, \implies \tilde{B}(t)^\top x_k = 0, \ \forall t \in [-k,k].$$

Now, let $k \in \mathbb{N}_+$. (Be careful that x_k is varying over time.) The above defines a bounded (by unity) sequence of vectors $\{x_k\}_{k=1}^{\infty}$ and it follows that $\exists$ a convergent subsequence $\{x_{k_j}\}_{j=1}^{\infty}$. (Sequential compactness theorem) Denote the limit as

$$x_0 = \lim_{j \to \infty} x_{k_j}.$$

Taking j sufficiently large $j \geq J_a$, $t_a \in [-k_j, k_j]$ for all $j \geq J_a$. Hence, $x_{k_j}^\top \tilde{B}(t_a) = 0$ for all $j \geq J_a$, thus

$$x_0^\top \tilde{B}(t_a) = 0.$$

Now, perform a change of coordinate with constant

$$P^{-1} := \begin{bmatrix} \vdots \\ x_0^{\mathsf{T}} \end{bmatrix},$$

in which the matrices $(\tilde{A}, \tilde{B}, \tilde{C})$ become

$$P^{-1}\tilde{B}(t) = \begin{bmatrix} \hat{B}_1(t) \\ 0_{1\times m} \end{bmatrix},\ C(t)P = \begin{bmatrix} \hat{C}_1(t) & \hat{C}_2(t) \end{bmatrix}.$$

It is easy to verify

$$H(t,\tau) = \hat{C}_1(t)\hat{B}_1(\tau),$$

and thus we get another realization $(0, \hat{B}_1(t), \hat{C}_1(t))$ with dimension $(n-1)$. This contradicts minimality of the given n-dimensional realization. Therefore, W_c is invertible. A similar procedure can be done for W_o. □

LTI Realization

Theorem 7 *(LTI Realization)* A weighting pattern $H(t, \tau)$ is realizable by an LTI model (A, B, C) iff H is realizable, continuously differentiable w.r.t. both t and τ, and

$$H(t, \tau) = H(t - \tau, 0), \quad \forall t, \tau.$$

In this case, $\exists$ a time-invariant minimal realization, which is controllable and observable.

Proof

($\Rightarrow$) For LTI systems, $H(t,\tau) = Ce^{A(t-\tau)}B$, and it is easy to verify $H(t,\tau) = H(t-\tau,0)$.

($\Leftarrow$) Pick a minimal realization $(0, B_t, C_t)$, where B_t and C_t are C^1-continuous, with $H(t,\tau) = C(t)B(\tau)$. For some $t_0 < t_1$,

$$W_c(t_0,t_1) = \int_{t_0}^{t_1} B_\tau B_\tau^\top d\tau \succ 0, \; W_o(t_0,t_1) = \int_{t_0}^{t_1} C_\tau^\top C_\tau d\tau \succ 0$$

We have $\frac{\partial}{\partial t}H(t,\tau) = -\frac{\partial}{\partial \tau}H(t,\tau)$, since $H(t,\tau) = H(t-\tau,0)$ and H is C^1. Therefore, $\forall t,\tau$

$$\dot{C}(t)B(\tau) = -C(t)\dot{B}(\tau) \implies \dot{C}(t)B(\tau)B(\tau)^\top + C(t)\dot{B}(\tau)B(\tau)^\top = 0.$$

Integrating w.r.t. τ yields

$$\begin{aligned}
&\dot{C}(t)W_c(t_0,t_1) + C(t)\int_{t_0}^{t_1}\dot{B}(\tau)B(\tau)^\top d\tau = 0,\ \forall t.\\
\implies\quad &\dot{C}(t) = C(t)\underbrace{\left[-\int_{t_0}^{t_1}\dot{B}(\tau)B(\tau)^\top d\tau W_c(t_0,t_1)^{-1}\right]}_{A}\\
\implies\quad &C(t) = C(0)e^{At}.
\end{aligned}$$

We get $H(t,\tau) = H(t-\tau,0) = C(t-\tau)B(0) = C(0)e^{A(t-\tau)}B(0)$, and an LTI realization is $(A, B(0), C(0))$. We do not change the dimension, so it is still minimal. ■

Examples

1. Consider the weighting pattern:

$$H(t,\tau) = e^{t+\tau}$$

   This is realizable, though the condition for time-invariant realizability clearly fails.

2. For the weighting pattern

$$H(t,\tau) = e^{-t^2+2t\tau-\tau^2}$$

   the LTI condition is easy to verify:

$$H(t-\tau,0) = e^{-(t-\tau)^2} = H(t,\tau).$$

   However, it takes a bit of thought even in this simple case to see that the weighting pattern is not realizable.

3. LTI Realization of Transfer Functions/Matrices

Now we consider the particular case of LTI realization of transfer functions/matrices. A model (A, B, C, D) has the transfer function

$$G(s) = C(sI - A)^{-1}B + D,$$

which is rational and proper.[1]

Theorem 8 The transfer function $G(s)$ admits a time-invariant realization $(A, B, C, 0)$ iff $G(s)$ is strictly proper and rational.

[1] Proper refers to a TF in which the degree of the numerator does not exceed ($\leq$) the degree of the denominator. For "$<$", we call it strictly proper.

Proof

($\Rightarrow$) The TF for a state-space model ($D = 0$) has all its elements strictly proper rational functions. Linear combinations of strictly proper rational functions are still strictly proper. (Rugh, Ch. 5)

($\Leftarrow$) Suppose $G_{ij}(s)$ is rational and strictly proper. Without loss of generality, we assume its denominator is monic, i.e., the coefficient of the highest power of s is unity. Let

$$d(s) = s^r + d_{r-1}s^{r-1} + \ldots + d_0 \tag{3}$$

be the monic least common multiple of these denominator polynomials:

$$d(s)G(s) = N_{r-1}s^{r-1} + \ldots + N_1 s + N_0.$$

We take

$$A = \begin{bmatrix} 0_m & I_m & \dots & 0_m \\ 0_m & 0_m & \dots & 0_m \\ \vdots & \vdots & \ddots & \vdots \\ 0_m & 0_m & \dots & I_m \\ -d_0 I_m & -d_1 I_m & \dots & -d_{r-1} I_m \end{bmatrix}, B = \begin{bmatrix} 0_m \\ \vdots \\ 0_m \\ I_m \end{bmatrix}$$

$$C = \begin{bmatrix} N_0 & N_1 & \cdots & N_{r-1} \end{bmatrix}$$

Let

$$Z(s) = (sI - A)^{-1} B$$

and partition $Z(s)$ into r blocks $Z_1(s), Z_2(s), \dots, Z_r(s)$. Multiplying by $(sI - A)$ gives

$$Z_{i+1}(s) = sZ_i(s), \quad i = 1, \dots, r-1$$

and $sZ_r(s) + d_0 Z_1(s) + d_1 Z_2(s) + \cdots + d_{r-1} Z_r(s) = I_m$.

Combining the above gives

$$Z_1(s) = \frac{1}{d(s)} I_m, \quad Z(s) = \frac{1}{d(s)} \begin{bmatrix} I_m \\ sI_m \\ \vdots \\ s^{r-1} I_m \end{bmatrix}$$

Finally, multiplying through by C yields

$$C(sI - A)^{-1}B = \frac{1}{d(s)} \left[N_0 + N_1 s + \ldots + N_{r-1} s^{r-1} \right] = G(s).$$

Thus, the above selection of (A, B, C) is a realization of $G(s)$. ■

Note: This specific realization is always controllable, but not necessarily observable (minimal).

Example 1: Find a Realization

Find a realization for the transfer matrix

$$G(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}.$$

This TF is not strictly proper, but we may take out the matrix D by

$$D = \lim_{s\to\infty} G(s) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{G}(s) = \frac{1}{(s+\frac{1}{2})(s+2)^2} \begin{bmatrix} -6(s+2)^2 & 3(s+\frac{1}{2})(s+2) \\ \frac{1}{2}(s+2) & (s+1)(s+\frac{1}{2}) \end{bmatrix}$$

$$d(s) = (s+\frac{1}{2})(s^2+4s+4) = s^3 + \frac{9}{2}s^2 + 6s + 2$$

$$N_0 = \begin{bmatrix} -24 & 3 \\ 1 & \frac{1}{2} \end{bmatrix},\ N_1 = \begin{bmatrix} -24 & \frac{15}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix},\ N_2 = \begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}.$$

Realization for SISO Systems

In the SISO case, with

$$G(s) = \frac{c_{n-1}s^{n-1} + \cdots + c_1 s + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0},$$

we can immediately give the control canonical form

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

and

$$y(t) = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \end{bmatrix} x(t).$$

4. Minimal Realization of Transfer Matrices

Markov Parameters

- In the LTI case, a minimal realization should be controllable and observable, since the uncontrollable/unobservable part in the Kalman decomposition do not appear.
- Note that

$$(sI - A)^{-1} = \frac{1}{s}\left(I - \frac{A}{s}\right)^{-1} = \sum_{i=0}^{\infty} s^{-(i+1)} A^i,$$

for $|s| > \rho(A)$, thus

$$C(sI - A)^{-1}B + D = D + \sum_{i=0}^{\infty} s^{-(i+1)} C A^i B$$

Definition 9 (Markov Parameters) Let $G(s)$ be a proper rational transfer function with

$$G(s) = \sum_{i=0}^{\infty} \frac{G_i}{s^i} = G_0 + G_1 \frac{1}{s} + G_2 \frac{1}{s^2} + \dots,$$

and the matrices $\{G_i\}_{i=0}^{\infty}$ are called the Markov parameters of G. If $G = C(sI - A)^{-1}B + D$, then for $i \geq 1$

$$G_0 = D, \quad G_i = CA^{i-1}B.$$

- The Markov parameters CA^iB of a state-space model $0 \leq i \leq n-1$ determine uniquely those for $i \geq n$, by **Cayley-Hamilton theorem**.
- They can be measured from the impulse response.

Calculation of Markov Parameters

A transfer matrix $G(s) = G_0 + \tilde{G}(s)$, with $\tilde{G}$ strictly proper rational.

$$\lim_{s\to\infty} s\tilde{G}(s) = G_1 \qquad [= G(0_+) = \lim_{t\to 0_+} G(t)]$$

$$G^{[1]}(s) := s\tilde{G}(s) - G_1 = \frac{G_2}{s} + \frac{G_3}{s^2} + \dots$$

$$\lim_{s\to\infty} sG^{[1]}(s) = G_2 \qquad [= \lim_{t\to 0^+} \frac{d}{dt}G(t)]$$

$$\text{etc.} \quad \lim_{t\to 0_+} \frac{d^i G(t)}{dt} = G_{i+1}$$

Time Domain

For $G(t) = Ce^{At}B + D\delta(t)$, D is the impulse part, and $\frac{d^i}{dt^i}G(t) = CA^i e^{At}B$ for $t > 0, i \geq 1$, so

$$CA^iB = \lim_{t\to 0_+} \frac{d^i}{dt^i}G(t) = G_{i+1}.$$

As a result, any two realizations of a transfer function/matrix must have the same Markov parameters. Clearly, D is uniquely defined via $\lim_{s\to\infty} G(s)$.

Consider two realizations:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \\ y = \tilde{C}\tilde{x} + \tilde{D}u \end{cases} \tag{4}$$

Recall that

$$\mathbb{O}\mathbb{C} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & \dots & A^{n-1}B \end{bmatrix} = \underbrace{\begin{bmatrix} CB & CAB & \dots & CA^{n-1}B \\ CAB & CA^2B & \dots & CA^nB \\ \vdots & \vdots & \ddots & \vdots \\ CA^{n-1}B & CA^nB & \dots & CA^{2n-2}B \end{bmatrix}}_{:=\mathcal{H}_n \quad \text{(Hankel Matrix)}}$$

Therefore, we have

$$\mathbb{O}\mathbb{C} = \tilde{\mathbb{O}}\tilde{\mathbb{C}}$$

Corollary 10 Two LTI system representations $R = (A, B, C, D)$ and $\tilde{R} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are realizations of the same transfer matrix $\hat{H}(s) \in \mathbb{C}^{n_o \times n_i}(s)$ iff one of the following equivalent statements holds:

1. R and $\tilde{R}$ have the same impulse response, i.e., $H(t) = \tilde{H}(t)$ for all $t \in \mathbb{R}$, where

$$H(t) = Ce^{At}B + D\delta(t), \quad \tilde{H}(t) = \tilde{C}e^{\tilde{A}t}\tilde{B} + \tilde{D}\delta(t);$$

2. $CA^iB = \tilde{C}\tilde{A}^i\tilde{B}$, for $i = 0, 1, 2, \ldots,$ and $D = \tilde{D}$.

Proposition 1 *For any realization* (A, B, C, D) *of* $\hat{H}(s) \in \mathbb{C}^{n_o \times n_i}(s)$,

$$\text{rank}(\mathcal{H}_l) = \text{rank}(\mathcal{H}_n), \quad \forall l \geq n,$$

and rank$(\mathcal{H}_n)$ *is independent of the given realization, where* $\mathcal{H}_j$ *is the Hankel matrix.*

Proof

From the Cayley-Hamilton Theorem, there exist nonsingular matrices L and R such that

$$L \begin{bmatrix} C \\ \vdots \\ CA^{n-1} \\ CA^n \\ \vdots \\ CA^{l-1} \end{bmatrix} = \begin{bmatrix} C \\ \vdots \\ CA^{n-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and

$$\begin{aligned}&\begin{bmatrix}B & AB & \cdots & A^{n-1}B & A^nB & \cdots & A^{l-1}B\end{bmatrix}R\\ =&\begin{bmatrix}B & AB & \cdots & A^{n-1}B & 0 & \cdots & 0\end{bmatrix}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{H}_l &= \mathcal{O}_l\mathcal{C}_l\\ &= L^{-1}\begin{bmatrix}\mathcal{O}_n & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathcal{C}_n & 0\end{bmatrix}R^{-1}\\ &= L^{-1}\begin{bmatrix}\mathcal{H}_n & 0\\ 0 & 0\end{bmatrix}R^{-1}.\end{aligned}$$

and therefore rank$(\mathcal{H}_l)$ = rank$(\mathcal{H}_n)$, $\forall l \geq n$. ∎

Note that we use the notations

$$\mathcal{O}_l = \begin{bmatrix}C\\ \vdots\\ CA^{l-1}\end{bmatrix},\ \mathcal{C}_l = \begin{bmatrix}B & \dots & A^{l-1}B\end{bmatrix}$$

MCMillan Degree

Definition 11 Define the *McMillan degree* of $\hat{H}(s)$ as

$$\delta_M := \max\{\text{rank}(\mathcal{H}_l) \mid l = 0, 1, \cdots\}.$$

The McMillan degree δ_M is the dimension of any minimal realization of $\hat{H}(s)$, s.t.

$$\delta_M = \text{rank}(\mathcal{H}_n) = \text{rank}(\mathbb{O}\mathbb{C}).$$

Minimal realization: Kalman & Ho

Theorem 12 *(Minimal realization: Kalman & Ho) Let* $\hat{H}(s) \in \mathbb{C}^{p\times m}(s)$ *be a proper transfer matrix with McMillan degree* δ_M*. Let* (A, B, C, D) *be an arbitrary realization of dimension* n*. Then,*

1. $\delta_M \leq n$;
2. $\delta_M = n$ *is equivalent to the pair* (A, B) *being controllable and the pair* (C, A) *being observable.*

Proof

For any realization (A, B, C, D), we have $\delta_M = \text{rank}(\mathbb{O}\mathbb{C})$. Hence by Sylvester's inequality, since $\mathbb{O}$ and $\mathbb{C}$ have n columns and n rows,

$$\text{rank}(\mathbb{O}) + \text{rank}(\mathbb{C}) - n \le \delta_M \le \min\{\text{rank}(\mathbb{O}), \text{rank}(\mathbb{C})\} \le n.$$

Therefore, 1) follows from the RHS. Since

$$(A, B) \text{ is controllable } \iff \text{rank}(\mathbb{C}) = n, \quad \text{and}$$
$$(C, A) \text{ is observable } \iff \text{rank}(\mathbb{O}) = n,$$

2) is also obvious. ■

Relation Between Two Minimal Realizations

- This Theorem shows that the Hankel matrix $\mathcal{H}_n = \mathbb{O}\mathbb{C}$ is *invariant* for all minimal realizations of a given transfer function.
- The McMillan degree δ_M is also the minimum number of integrators needed to simulate the given transfer matrix $\hat{H}(s)$.
- The next theorem establishes that minimal realizations of the same transfer function are *algebraically equivalent*.

Theorem 13 *Any two minimal realizations are related by a unique change of coordinates.*

Proof

Consider the above two realizations, with $\mathbb{O}\mathbb{C} = \tilde{\mathbb{O}}\tilde{\mathbb{C}}$. $\mathbb{C}$ is full row rank, so $\mathbb{C}^r := \mathbb{C}^\top(\mathbb{C}\mathbb{C}^\top)^{-1}$ satisfies $\mathbb{C}\mathbb{C}^r = I_n$. Similarly, $\mathbb{O}$ is full column rank, so $\mathbb{O}^l = (\mathbb{O}^\top\mathbb{O})^{-1}\mathbb{O}^\top$ satisfies $\mathbb{O}^l\mathbb{O} = I_n$.

Define $T = \tilde{\mathbb{C}}\mathbb{C}^r$, which is invertible, i.e.

$$\mathbb{O}^l\tilde{\mathbb{O}}T = \mathbb{O}^l\tilde{\mathbb{O}}\tilde{\mathbb{C}}\mathbb{C}^r = \mathbb{O}^l\mathbb{O}\mathbb{C}\mathbb{C}^r = I \implies T^{-1} = \mathbb{O}^l\tilde{\mathbb{O}}.$$

We also have

$$\mathbb{O}\mathbb{C} = \tilde{\mathbb{O}}\tilde{\mathbb{C}} \implies \mathbb{O} = \tilde{\mathbb{O}}\tilde{\mathbb{C}}\mathbb{C}^r = \tilde{\mathbb{O}}T \implies \mathbb{O} = \tilde{\mathbb{O}}T$$

$$\mathbb{O}\mathbb{C} = \tilde{\mathbb{O}}\tilde{\mathbb{C}} \implies \mathbb{C} = \mathbb{O}^l\tilde{\mathbb{O}}\tilde{\mathbb{C}} = T^{-1}\tilde{\mathbb{C}} \implies \tilde{\mathbb{C}} = T\mathbb{C}$$

Invoking the definition of observability and controllability matrices, we have

$$\tilde{B} = TB, \quad \tilde{C} = CT^{-1}.$$

From the equivalence of Markov parameters, we have

$$\begin{aligned}
\mathbb{O}A\mathbb{C} = \tilde{\mathbb{O}}\tilde{A}\tilde{\mathbb{C}} \quad &\Longrightarrow \quad \tilde{\mathbb{O}}^l\mathbb{O}A\mathbb{C}\mathbb{C}^r = \tilde{A}\tilde{\mathbb{C}}\mathbb{C}^r \\
&\Longrightarrow \quad \tilde{\mathbb{O}}^l\mathbb{O}A = \tilde{A}\tilde{\mathbb{C}}\mathbb{C}^r \\
&\Longrightarrow \quad TA = \tilde{A}T \\
&\Longrightarrow \quad A = T^{-1}\tilde{A}T.
\end{aligned}$$

We have related two realizations via a change of coordinate. ■

Gilbert's Method for Minimal Realizations ⋆

Gilbert's Realization is a particular minimal realization which can be obtained directly from a transfer function matrix $G(s)$. However, this realization is possible only when each entry of $G(s)$ has distinct poles.

1. Expand each entry of $G(s)$ into partial fractions and form

$$G(s) = D + \frac{R_1}{s - \lambda_1} + \cdots + \frac{R_q}{s - \lambda_q}, \qquad R_i \in \mathbb{R}^{p \times m},$$

2. Total size of realization is

$$n^* = \sum_i \text{Rank}(R_i), \quad r_i = \text{rank}\,(R_i)$$

3. Find B_i and C_i so that

$$C_i B_i = R_i \quad \text{where } C_i \in \mathbb{R}^{p \times r_i},\ B_i \in \mathbb{R}^{r_i \times m}.$$

The Gilbert's Realization is

$$A = \begin{bmatrix} \lambda_1 I_{r_1} & & \\ & \ddots & \\ & & \lambda_q I_{r_q} \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ \vdots \\ B_q \end{bmatrix},\ C = \begin{bmatrix} C_1 & \dots & C_q \end{bmatrix},\ D.$$

- The procedure is complex if λ_i's are complex, and we need to work a bit more to get real roots.
- This realization is controllable and observable.

Example 2

Find a minimal realization of the following transfer function

$$G(s) = \begin{bmatrix} \frac{1}{(s-1)(s-2)} & \frac{1}{(s-2)(s-3)} \\ \frac{1}{(s-2)(s-3)} & \frac{1}{(s-1)(s-2)} \end{bmatrix}$$

Solution. Since all entries of $G(s)$ have simple poles, we can use Gilbert realization.

$$\begin{aligned} G(s) &= \begin{bmatrix} \frac{-1}{s-1} + \frac{1}{s-2} & \frac{-1}{s-2} + \frac{1}{s-3} \\ \frac{-1}{s-2} + \frac{1}{s-3} & \frac{-1}{s-1} + \frac{1}{s-2} \end{bmatrix} \\ &= \frac{1}{s-1}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{s-2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{s-3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \frac{1}{s-1}\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{C_1}\underbrace{I_2}_{B_1} + \frac{1}{s-2}\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{C_2}\underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{B_2} + \frac{1}{s-3}\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{C_3}\underbrace{I_2}_{B_3} \end{aligned}$$

Solution

Therefore,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 0 \end{bmatrix}.$$

5. Balanced Realization

Balanced Realization

- Consider a stable and minimal realization of form

$$\dot{x} = Ax + Bu, \quad y = Cx.$$

  We want to obtain the balanced realization, which is a type of minimal realization with diagonal gramians.

- Then,

$$AW_c + W_c A^\top = -BB^\top$$
$$A^\top W_o + W_o A = -C^\top C,$$

  where the controllability Gramian W_c and the observability Gramian W_o are positive definite.

Theorem 14 Suppose that two different state space realizations (A, B, C) and $(\hat{A}, \hat{B}, \hat{C})$ are minimal and equivalent. Let $W_c W_o$ and $\hat{W}_c \hat{W}_o$ be the products of their controllability Gramian and observability Gramian, respectively. Then, $W_c W_o$ and $\hat{W}_c \hat{W}_o$ are similar and have positive eigenvalues.

Proof.

Since we have proven that two minimal realizations can be related by a unique change of variables, write

$$\hat{A} = T^{-1} A T, \quad \hat{B} = T^{-1} B, \quad \hat{C} = C T.$$

Then

$$\hat{A} \hat{W}_c + \hat{W}_c \hat{A}^T = -\hat{B} \hat{B}^T$$

yields

$$T^{-1}AT\hat{W}_c + \hat{W}_c T^\top A^\top T^{-\top} = -T^{-1}BB^T T^{-\top}$$
$$AT\hat{W}_c T^\top + T\hat{W}_c T^\top A^\top = -BB^\top$$
$$A\underbrace{T\hat{W}_c T^\top}_{W_c} + \underbrace{T\hat{W}_c T^\top}_{W_c} A^\top = -BB^\top.$$

Thus, we have

$$W_c = T\hat{W}_c T^\top,$$

and similarly,

$$W_o = T^{-\top}\hat{W}_o T^{-1}.$$

Now,

$$W_c W_o = T\hat{W}_c T^\top T^{-\top}\hat{W}_o T^{-1} = T\hat{W}_c\hat{W}_o T^{-1},$$

which implies that $W_c W_o$ and $\hat{W}_c\hat{W}_o$ are similar. To complete the proof, we need the following lemma.

Lemma 15 For every real symmetric matrix X, $\exists$ an orthogonal matrix Q s.t. $X = Q^\top D Q$ where D is a diagonal matrix with $\lambda_i(X)$'s being real.

Note W_c is p.d, so we write

$$W_c = Q^\top D^{\frac{1}{2}} D^{\frac{1}{2}} Q =: R^\top R$$

where Q is orthogonal, i.e., $Q^{-1} = Q^\top$, and $R = D^{\frac{1}{2}} Q$. Consider

$$\begin{aligned} \det(sI - W_c W_o) &= \det(sI - R^\top R W_o) = \det(R^\top)\det(sR^{-\top} - RW_o) \\ &= \det(sR^{-\top} - RW_o)\det(R^\top) \\ &= \det(sI - RW_o R^\top) \end{aligned}$$

implying $W_c W_o$ and $RW_o R^\top$ have the same eigenvalues. Here, note that $RW_o R^\top$ is p.d.; therefore, $\lambda_i(W_c W_o) > 0$. ■

Hankel Singular Values

Definition 16 We define the Hankel singular values, σ_i, as the square roots of the eigenvalues of $W_c W_o$, i.e.

$$\sigma_i = \sqrt{\lambda_i(W_c W_o)}.$$

- This defines a new set of *invariant* parameters.
- They tell us which modes dominate the I/O behavior.
- Very important in model reduction.

Balanced Realization

Theorem 17 For any minimal realization (A, B, C), $\exists$ a similarity transformation s.t. W_c and W_o of its equivalent s.s. realization satisfy

$$\hat{W}_c = \hat{W}_o = \Sigma.$$

Such an equivalent realization is called balanced realization.

Proof.

Recall the expression $RW_oR^\top$ where

$$W_c = R^\top R, \quad R = D^{\frac{1}{2}}Q.$$

Since $RW_oR^\top$ is symmetric, we can write

$$RW_oR^\top = U\Sigma^2U^\top,$$

cont'd.

where U is orthogonal. Then we can write

$$U^\top R W_o R^\top U = \Sigma^2$$

and with $W_o = T^{-\top}\hat{W}_o T^{-1}$

$$\underbrace{\Sigma^{-\frac{1}{2}}U^\top R}_{T^{-\top}} W_o \underbrace{R^\top U \Sigma^{-\frac{1}{2}}}_{T^{-1}} = \Sigma =: \hat{W}_o.$$

Similarly, with $W_c = T\hat{W}_c T^\top$

$$\begin{aligned}\underbrace{\Sigma^{\frac{1}{2}}U^\top R^{-\top}}_{T} W_c \underbrace{R^{-1}U\Sigma^{\frac{1}{2}}}_{T^\top} &= \Sigma^{\frac{1}{2}}U^\top R^{-\top}R^\top R R^{-1}U\Sigma^{\frac{1}{2}} \\ &= \Sigma =: \hat{W}_c.\end{aligned}$$

■

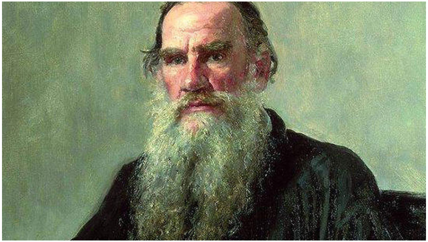
Balanced Realization

- Every system $G(s) \in RH_\infty$ has a minimal balanced realization.
- In the balanced realization, the controllability and observability Gramians are equal. Hence strongly controllable states are also strongly observable, and weakly controllable states are also weakly observable

$$\hat{W}_c = \hat{W}_o = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}$$

- We can always choose the ordering so that $\sigma_i \geq \sigma_{i+1}$.
- Hence, we might expect that removing the weakly observable and weakly controllable states would result in a low model-reduction error.

Linear vs Nonlinear



"All happy families (linear systems) are alike, every unhappy family (nonlinear system) is unhappy (nonlinear) in its own way."

Leo Tolstoy

Sources and Further Reading

- Realization concepts for time-varying systems: The notions of weighting patterns, realizability, nonuniqueness of realizations, and factorization-based existence conditions draw primarily on Rugh.
- Minimal realizations: This follows standard treatments in Rugh, (Callier and Desoer), and (Antsaklis and Michel).
- Realization of rational transfer matrices: This draws mainly on Rugh and (Antsaklis and Michel).
- Markov parameters and Hankel matrices: (Callier and Desoer) and (Antsaklis and Michel).