Generation of Oscillations in Nonlinear Systems: Applications to Nonholonomic and Underactuated Mechanical Systems

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Contents

C1. Oscillation Generation via

- Immersion and Invariance (I&I)
  (R. Ortega, B. Yi, J. Romero and A. Astolfi, *Int. J. of Robust and Nonlinear Control*'19)
- Interconnection and Damping Assignment (IDA) Passivity-based Control (PBC)
  (B. Yi, R. Ortega and D. Wu, *Automatica'19)*
- Mexican Sombrero Energy Assignment (MSEA)
- Energy Pumping and Damping (EPD)

C2. Application to Mechanical Systems

- Smooth, Time-invariant Regulation of Nonholonomic Systems
  (B. Yi and R. Ortega, *Int. J. of Robust and Nonlinear Control*'19, under review)
- Path Following of Underactuated Mechanical Systems
  (B. Yi, R. Ortega, I.R. Manchester and H. Siguerdidjane, arXiv'20)

Formulation of the Orbital Stabilization Problem

Problem Formulation

Given the system $\dot{x} = f(x) + g(x)u$, $x \in \mathbb{R}^n, u \in \mathbb{R}^m$, generate asymptotically orbitally stable periodic solutions

$$X(t) = X(t+T), \ \forall t \geq 0.$$

That is, find $\hat{u}(x)$ such that the closed-loop system

$$\dot{x} = f(x) + g(x)\hat{u}(x) =: f_{\mathtt{cl}}(x),$$

satisfies

$$\dot{X}(t) = f_{\mathtt{cl}}(X(t)),$$

and

$$\mathcal{A} := \{x \in \mathbb{R}^n \mid x = X(t), \ 0 \leq t \leq T\},$$

is an attractive set.

Orbital vs Set Stabilization

- Asymptotic set stabilization is satisfied ensuring

$$\liminf \|x(t)\|_{\mathcal{A}} = 0,$$

where $\|x\|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$.

- This does not ensure that the desired periodic motion is generated. Indeed, if the set

$$\mathcal{O} := \{x \in \mathbb{R}^n \mid \|x\|_{\mathcal{A}} = 0\}$$

contains equilibrium points of the closed-loop dynamics the periodic motion is not generated.

- We have to ensure that

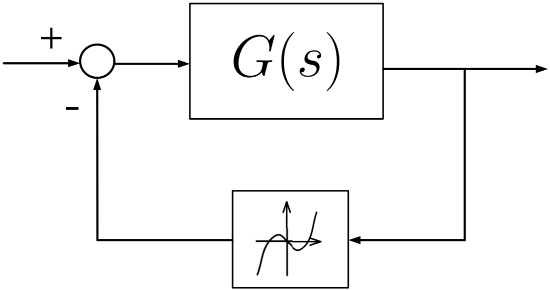
$$f_{\mathtt{cl}}(x)|_{x \in \mathcal{O}} \neq 0.$$

Motivations

- Applications:
  - Mechanical systems (walking and running robots, path following)
  - Rotating electromechanical systems (motors, AC or resonant power converters)
  - Biology (oscillation mechanisms)

- Analysis well studied, but only a few constructive synthesis tools:
  - Virtual holonomic constraints (VHC) method for mechanical systems (Shyriaev, Maggiore, Grizzle,...).
    (a certain subspace rendered attractive and invariant, leading to a projected dynamics that behaves as oscillators.)
  - In (Stan/Sepulchre) construct passive oscillators for Lure dynamical systems using "sign-indefinite" feedback static mappings—intrinsically local (central manifold).

Lure System with Sign-indefinite Feedback



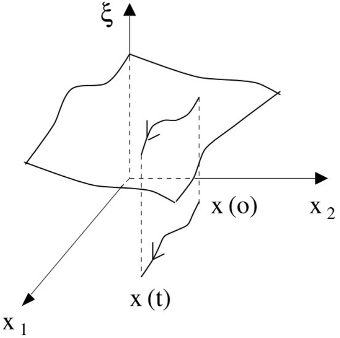
"Sign" of feedback changes as a function of the output value.

Contributions

- Objectives
  - Show that I&I and IDA-PBC can be adapted for orbital stabilization of general NL systems;
  - apply it for nonholonomic systems.
  - apply to path following of underactuated systems
- VHC is a particular case of the I&I technique used for equilibrium stabilization in (Astolfi/Ortega'03), and later extended for observer design and adaptive control in (AKO'07, SV book).
- A main drawback of VHC and I&I methods is that the steady-state behavior cannot be fixed a priori, but depends on the initial states
- Overcome with IDA in two versions
  - EPD, which contains (Stan/Sepulchre) as a particular case.
  - MSEA method.
  - Establish connections between them.

Orbital Stabilization via I&I

Immersion and Invariance Principle



- Immerse the plant dynamics into a lower-order target system

$$\dot{\xi} = \alpha(\xi), \xi \in \mathbb{R}^p, p < n,$$

- Creating an attractive and invariant manifold

$$\begin{aligned} \mathcal{M} &= \{x \in \mathbb{R}^n \mid x = \pi(\xi),\ \xi \in \mathbb{R}^p\} \\ &= \{x \in \mathbb{R}^n \mid \phi(x) = 0\} \end{aligned}$$

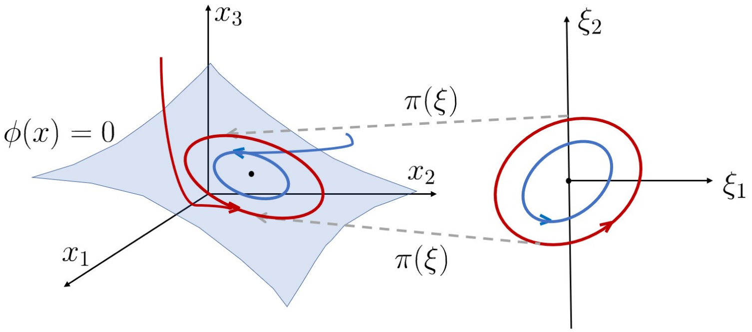
- Whose "zero dynamics" (wrt output $\phi(x)$) is the target system.
- For orbital stabilization select it with a periodic orbit.

Pictorial Representation

Plot of the manifold

$$\mathcal{M} = \{x \in \mathbb{R}^n \mid x = \pi(\xi),\ \xi \in \mathbb{R}^p\} = \{x \in \mathbb{R}^n \mid \phi(x) = 0\}$$

target dynamics trajectory $\dot{\xi} = \alpha(\xi)$ and immersion mapping $\pi(\xi)$



Main Orbital Stabilization Result of I&I

Consider $\dot{x} = f(x) + g(x)u$. Assume there are mappings

$$\alpha : \mathbb{R}^p \to \mathbb{R}^p,\ \pi : P\mathbb{R}^p \to \mathbb{R}^n,\ \phi : \mathbb{R}^n \to \mathbb{R}^{n-p},\ v : \mathbb{R}^{n\times(n-p)} \to \mathbb{R}^m,$$

such that:

- (H1) (Target oscillator) The dynamical system

$$\dot{\xi} = \alpha(\xi) \tag{1}$$

has non-trivial, periodic solutions $\xi_\star(t) = \xi_\star(t+T),\ \forall t \geq 0$, (parameterized by $\xi(0)$.)

- (H2) (Immersion condition) For all $\xi \in \mathbb{R}^p$

$$g^\perp(\pi(\xi))\left[f(\pi(\xi)) - \nabla\pi^\top(\xi)\alpha(\xi)\right] = 0.$$

where $g^\perp : \mathbb{R}^n \to \mathbb{R}^{n-m}$ is a full-rank left-annihilator of $g(x)$.

- (H3) (Implicit manifold) The following set identity holds

$$\mathcal{M} := \{x \in \mathbb{R}^n \mid \phi(x) = 0\} = \{x \in \mathbb{R}^n \mid x = \pi(\xi),\ \xi \in \mathbb{R}^p\}.$$

cont'd

- (H4) (Attractivity and boundedness) All the trajectories of the system

$$
\begin{aligned}
\dot{z} &= \nabla\phi^\top(x)[f(x) + g(x)v(x,z)] \\
\dot{x} &= f(x) + g(x)v(x,z)
\end{aligned}
$$

with $z(0) = \phi(x(0))$ and $v(\pi(\xi), 0) = c(\pi(\xi))$, where

$$
c(\pi(\xi)) := [g^\top(\pi(\xi))]^\dagger \left\{ \nabla\pi^\top(\xi)\alpha(\xi) - f(\pi(\xi)) \right\},
$$

are bounded and satisfy $\lim_{t\to\infty} z(t) = 0$.
Then, the system

$$
\dot{x} = f(x) + g(x)v(x, \phi(x)) =: f_{\mathtt{cl}}(x)
$$

ensures $x_\star(t) = \pi(\xi_\star(t))$ is orbitally attractive.

Sketch of the Proof

- Set invariance The set $\mathcal{M} = \{x \in \mathbb{R}^n \mid \phi(x) = 0\}$ is invariant with respect to $\dot{x} = f_{\mathtt{cl}}(x)$ if and only if

$$\begin{aligned}
\dot{\phi}(x)|_{x\in\mathcal{M}} = 0 \Leftrightarrow \dot{x}|_{x=\pi(\xi)} &= \dot{\pi}(\xi) \\
\Leftrightarrow\ & f(\pi(\xi)) + g(\pi(\xi))c(\pi(\xi)) = \nabla\pi^\top(\xi)\alpha(\xi) \\
\Leftrightarrow\ & \begin{bmatrix} g^\perp(\pi(\xi))\left[f(\pi(\xi)) - \nabla\pi^\top(\xi)\alpha(\xi)\right] = 0 \\ c(\pi(\xi)) = g^\dagger(\pi(\xi))\left\{\nabla\pi^\top(\xi)\alpha(\xi) - f(\pi(\xi))\right\} \end{bmatrix}
\end{aligned}$$

- Set attractivity

$$\begin{bmatrix} z(0) = \phi(x(0)) \text{ and } v(\pi(\xi),0) = c(\pi(\xi)) \\ \dot{z} = \dot{\phi} = \nabla\phi^\top(x)[f(x) + g(x)v(x,z)] \end{bmatrix} \Rightarrow z(t) = \phi(x(t)).$$

Hence, $\lim_{t\to\infty} z(t) = 0$ implies $\mathcal{M}$ attractive.

Remarks

- Main difficulties
  - Solving the PDE

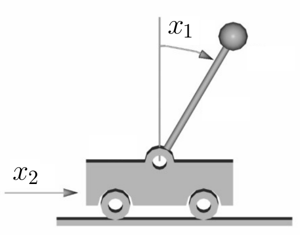
$$g^{\perp}(\pi(\xi))\left[f(\pi(\xi)) - \nabla\pi^{\top}(\xi)\alpha(\xi)\right] = 0.$$

  - Proving boundedness and convergence of $z(0) \to 0$ of

$$\begin{aligned}\dot{z} &= \nabla\phi^{\top}(x)[f(x) + g(x)v(x,z)]\\ \dot{x} &= f(x) + g(x)v(x,z)\end{aligned}$$

- Main drawback The periodic orbit depends on the ICs.

Example: Cart-pendulum System



- Model

$\dot{x}_1 = x_3$

$\dot{x}_2 = x_4$

$\dot{x}_3 = a_1 \sin(x_1) - a_2 \cos(x_1) u$

$\dot{x}_4 = u.$

- Control objective: With $x_1(0) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, oscillate around $x_1 = 0$.
- Target dynamics: undamped mechanical system

$$\dot{\xi}_1 = \xi_2,\ \dot{\xi}_2 = \alpha_2(\xi_1),$$

- Total energy function

$$\mathcal{H}_\xi(\xi) := \frac{1}{2}\xi_2^2 - \int_0^{\xi_1} \alpha_2(s) ds.$$

- Potential energy has minimum at zero: $\alpha_2(0) = 0, \alpha_2'(0) < 0 \Rightarrow \mathcal{H}_\xi = c$ are periodic orbits $\Rightarrow$ (H1).

Assumptions (H2) and (H3)

- Propose $\pi(\xi_1) := \mathrm{col}(\pi_1(\xi_1), \pi_2(\xi_1), \pi_1'(\xi_1)\xi_2, \pi_2'(\xi_1)\xi_2)$
- PDE of (H2)

$$
\begin{aligned}
& a_1 \sin(\pi_1(\xi_1)) - a_2 \cos(\pi_1(\xi_1)) \Big[\pi_2''(\xi_1)\xi_2^2 + \pi_2'(\xi_1)\alpha_2(\xi_1)\Big] \\
& = \pi_1''(\xi_1)\xi_2 + \pi_1'(\xi_1)\alpha_2(\xi_1) \;\Rightarrow\; \pi_1''(\xi_1) = \pi_2''(\xi_1) = 0
\end{aligned}
$$

- Propose $\pi_1(\xi) = \xi_1,\ \pi_2(\xi_1) = k\xi_1$ yielding

$$
\phi(x) = \begin{bmatrix} -k & 1 & 0 & 0 \\ 0 & 0 & -k & 1 \end{bmatrix} x, \qquad \text{(H3)}
$$

$$
\alpha_2(\xi_1) = \frac{a_1 \sin(\xi_1)}{1 + k a_2 \cos(\xi_1)},\ c(\pi(\xi)) = \frac{k a_1 \sin(\xi_1)}{1 + k a_2 \cos(\xi_1)}.
$$

- The minimum conditions impose $-\frac{1}{a_2} > k \Rightarrow$ upper-half plane!

Assumption H4

- Off-the-manifold dynamics

$$\dot{z}_1 = z_2,\ \dot{z}_2 = -ka_1 \sin(x_1) + [1 + ka_2 \cos(x_1)]v(x,z).$$

- Choose the control law

$$v(x,z) = \frac{1}{1 + ka_2 \cos(x_1)} \left[-\gamma_1 z_2 - \gamma_2 z_1 + ka_1 \sin(x_1)\right], \gamma_i > 0,$$

satisfies $v(\pi(\xi), 0) = c(\pi(\xi))$ and

$$\ddot{z}_1 + \gamma_1 \dot{z}_1 + \gamma_2 z_1 = 0.$$

- Closed-loop x dynamics, with $\epsilon_t \to 0$ (exp),

$$\ddot{x}_1 = \frac{a_1 \sin(x_1) - a_2 \cos(x_1)\epsilon_t}{1 + ka_2 \cos(x_1)},\ \ddot{x}_2 = \frac{a_1 \sin(x_1) + \epsilon_t}{1 + ka_2 \cos(x_1)},$$

verifies—after a painful analysis—boundedness for large γ_i!.

Orbital Stabilization via IDA-PBC

IDA–PBC for Equilibrium Stabilization

(Viewed as port-Hamiltonian (pH) system matching)

Consider $\dot{x} = f(x) + g(x)u$. Assume there are matrices

$$g^{\perp}(x), \quad \mathcal{J}(x) = -\mathcal{J}^{\top}(x), \quad \mathcal{R}(x) = \mathcal{R}^{\top}(x) \geq 0,$$

where

$$g^{\perp}(x)g(x) = 0$$

and $g^{\perp}(x)$ full rank, and a function $H(x)$, that verify the PDE

$$g^{\perp}(x)f(x) = g^{\perp}(x)[\mathcal{J}(x) - \mathcal{R}(x)]\nabla H \qquad \text{(ME)}$$

Define

$$\hat{u}(x) = [g^{\top}(x)g(x)]^{-1}g^{\top}(x)\{[\mathcal{J}(x) - \mathcal{R}(x)]\nabla H - f(x)\},$$

cont'd

- The closed–loop system with $u = \hat{u}(x)$ takes a pH form

$$\dot{x} = [\mathcal{J}(x) - \mathcal{R}(x)]\nabla H.$$

- Hence,

$$\dot{H} = (\nabla H)^\top \mathcal{R}(x) \nabla H \leq 0$$

- If $x_\star \in \{x \mid g^\perp(x) f(x) = 0\}$—an assignable equilibrium—and

$$x_\star = \arg\min H(x)$$

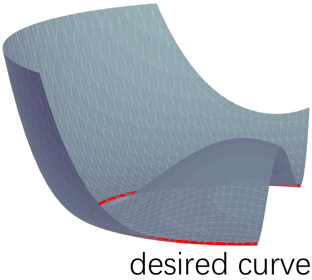
then it is a stable equilibrium.

- It is asymptotically stable if the "output" $\mathcal{R}(x)\nabla H(x)$ is detectable. That is

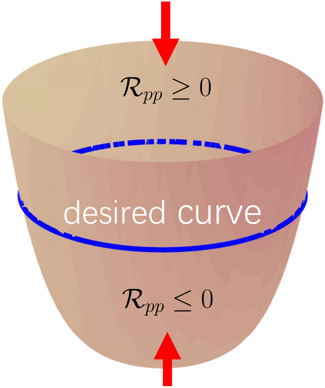
$$\mathcal{R}(x(t))\nabla H(x(t)) \equiv 0 \;\Rightarrow\; \lim_{t\to\infty} x(t) = x_\star.$$

MSEA and EPDI IDA-PBC: Pictorial Representation

- MSEA generates the orbit assigning it as the minimum of $H(x)$.



- EPDI generates the orbit with a sign-dependent damping $R(x)$.



A Lesson from Engineers

Field-oriented Control of Induction Motors

- Model of current-fed IMs, in rotating frame:

$$\dot{\lambda} = -R\lambda + Rv$$
$$\dot{\omega} = v^\top \mathbb{J}\lambda, \quad \mathbb{J} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

- Control objective $\beta(t) \to \beta_\star$, $\omega(t) \to \omega_\star$, where $|\lambda| = \beta$.
- In polar coordinates (β, ρ) as

$$\dot{\beta} = -R\beta + Ri_d,\ \dot{\rho} = \frac{R}{\beta} i_q,\ \dot{\omega} = \beta i_q,$$

where we have defined col$(i_d, i_q) := e^{-\mathcal{J}\rho} v$

- Famous direct FOC (Blaschke'72)

$$v = e^{\mathcal{J}\rho} \begin{bmatrix} \beta_\star \\ \frac{k}{\beta_\star}(\omega_\star - \omega) \end{bmatrix},\ k > 0$$

Field-oriented Control is Orbitally Stabilizing

- Model in fixed frame:

$$\dot{\psi}_r = -R\psi_r + \omega\mathbb{J}\psi_r + Ru$$
$$\dot{\omega} = u^\top \mathbb{J}\psi_r,$$

where

$$\psi_r := e^{\mathbb{J}\theta}\lambda, \quad u := e^{\mathbb{J}\theta}v, \quad \dot{\theta} = \omega,$$

- Control objective: orbital stabilization

$$\mathcal{A} := \{x \in \mathbb{R}^3 | \ |\psi_r| = \beta_\star, \ \omega = \omega_\star\},$$

- Define $x = \text{col}(x_p, x_\ell)$ and $x_p := \psi_r$, $x_\ell := \omega$.

Field-oriented Control is an MSEA IDA-PBC

- The closed-loop is an MSEA pH system

$$\dot{x} = [\mathcal{J}(x) - \mathcal{R}(x)]\nabla H.$$

with

$$\mathcal{R}(x) = \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & \frac{k}{\beta_\star}|x_p| \end{bmatrix}, \ \mathcal{J}(x) \quad = \begin{bmatrix} 0 & -\frac{x_\ell}{|x_p|-\beta_\star}|x_p| & \frac{kR}{\beta_\star}\frac{x_2}{|x_p|} \\ * & 0 & -\frac{kR}{\beta_\star}\frac{x_1}{|x_p|} \\ * & * & 0 \end{bmatrix}$$

- Energy function

$$H(x) = \frac{1}{2}(|x_p| - \beta_\star)^2 + \frac{1}{2}(x_\ell - \omega_\star)^2,$$

has a Mexican sombrero-like shape.

Field-oriented Control is an EPDI IDA-PBC

- The closed-loop is an EPDI pH system with

$$\mathcal{R}(x) = \begin{bmatrix} R(|x_p| - \beta_\star) & 0 & 0 \\ 0 & R(|x_p| - \beta_\star) & 0 \\ 0 & 0 & \frac{k}{\beta_\star}|x_p| \end{bmatrix}$$

$$\mathcal{J}(x) = \begin{bmatrix} 0 & -\omega|x_p| & \frac{kR}{\beta_\star}\frac{x_2}{|x_p|} \\ * & 0 & -\frac{kR}{\beta_\star}\frac{x_1}{|x_p|} \\ * & * & 0 \end{bmatrix},$$

with energy pumping and damping injection.

- Energy function

$$\mathcal{H}(x) = \frac{1}{2}|x_p|^2 + \frac{1}{2}(\omega - \omega_\star)^2.$$

Orbital Stabilization via MSEA

Orbital Stabilization Problem Formulation

- Particular case of periodic motion: $x := \text{col}(x_p, x_\ell)$, with $x_p \in \mathbb{R}^2$, $x_\ell \in \mathbb{R}^{n-2}$. Partition the matrices

$$\begin{bmatrix} (\cdot)_{pp} & (\cdot)_{p\ell} \\ (\cdot)_{\ell p} & (\cdot)_{\ell\ell} \end{bmatrix} \sim \begin{bmatrix} \mathbb{R}^{2\times 2} & \mathbb{R}^{2\times(n-2)} \\ \mathbb{R}^{(n-2)\times 2} & \mathbb{R}^{(n-2)\times(n-2)} \end{bmatrix}.$$

- The set to be orbitally stabilized is

$$\mathcal{A} = \{x_p \in \mathbb{R}^2 \mid \Phi(x_p) = 0\} \cup \{x_\ell^*\},$$

with $\nabla\Phi(x_p) \neq 0$—to ensure its a Jordan curve.

- To ensure the periodic motion is generated

$$f_{\mathtt{cl}}(x)|_{x\in\mathcal{O}} \neq 0,\ \mathcal{O} := \{x \in \mathbb{R}^n \mid \|x\|_{\mathcal{A}} = 0\}$$

Main Result

(A1) (pH Matching) There are $\mathcal{J}(x) = -\mathcal{J}^\top(x)$, $\mathcal{R}(x) \geq 0$ and $H_0 : \mathbb{R}^{n-1} \to \mathbb{R}$ verifying

$$\arg\min \; H_0(x_0, x_\ell) = (0, x_\ell^*) \quad \text{(isolated)},$$

solutions of the ME PDE, with $H(x) := H_0(\Phi(x_p), x_\ell)$.
(A2) (Attractivity) $\mathcal{A}$ is the largest invariant set in the set

$$\{x \in \mathbb{R}^n | \nabla^\top H(x)\mathcal{R}(x)\nabla H(x) = 0\} \cap B_\varepsilon(\mathcal{A}).$$

(A3) (Orbit existence) The (1,2)-element of $\mathcal{J}(x)$ verifies

$$\mathcal{J}_{(1,2)}(x) = \left.\frac{c(x)}{\nabla_{x_0} H_0(x_0, x_\ell)}\right|_{x_0 = \Phi(x_p)}$$

for some $0 < |c(x)| < \infty$, $\forall x \in \mathcal{A}$.
Then, the system with $u = \hat{u}(x)$ is asymptotically orbitally stable.

Key Step of the Proof: There are no Equilibria in $\mathcal{A}$

We will prove that $\dot{x}_p \neq 0$ for $x \in \mathcal{A}$. In $\mathcal{A}$ we have $\dot{x}_\ell = 0$ and:

$$\begin{aligned}
\dot{x}_p &= \mathcal{J}_{pp}(x)\nabla_{x_p} H(x) \\
&= \mathcal{J}_{pp}(x)\nabla_{x_0} H_0(x_0, x_\ell)\nabla\Phi(x_p) \\
&= \begin{bmatrix} 0 & \mathcal{J}_{(1,2)}(x) \\ -\mathcal{J}_{(1,2)}(x) & 0 \end{bmatrix} \nabla_{x_0} H_0(x_0, x_\ell)\nabla\Phi(x_p) \\
&= \begin{bmatrix} 0 & c(x) \\ -c(x) & 0 \end{bmatrix} \nabla\Phi(x_p),
\end{aligned}$$

Hence $\dot{\Phi} = 0$ and $\Phi(x_p) = 0$ is invariant. Compute the 1-norm

$$\|f_{\mathtt{cl}}(x)\|_1 = |c(x)| \|\nabla\Phi(x_p)\|_1 > 0, \quad \forall x \in \mathcal{A}.$$

Discussions

- A result similar to MSEA IDA-PBC exists for EPD IDA-PBC.
- Condition under which both methods generate the same control.
- Applicable to nonholonomic systems, *i.e.*, point regulation with a smooth time-invariant feedback. (complying Brockett's necessary condition)
- Partial results on path following, *e.g.*, for ships.

Smooth, Time-invariant Regulation of Nonholonomic Systems via EPD

Background and Control Objective

- Nonholonomic systems cannot by asymptotically stabilized with a continuous, time-invariant (static or dynamic), state-feedback.
- Time-varying (Pomet'92, Jiang, et al.'01), discontinuous (Astolfi'96, Fujimoto, et al.'12) and switching control methods (Liberzon), considered.
- In (Escobar, et al. '98) the EPD (FOC) method applied to nonholonomic integrator, but the controller is not-globally defined.
- This result is the inspiration for the popular transverse function approach (Morin/Samson'03,'09).
- Objectives:
  - Find a globally defined, smooth, time-invariant state-feedback that ensures $x(t) \to 0$ or $H(x(t)) \to H_\star$.
  - Show the transient performance improvement.

Application of EPD IDA-PBC

- Consider (completely) nonholonomic systems

$$\begin{aligned} \dot{x} &= S(x)u \\ 0 &= A^\top(x)\dot{x}, \quad \text{rank}\,\{A(x)\} = n - m \qquad \text{(NHS)} \end{aligned}$$

- IDA-PBC formulation: Partition $x = \text{col}(x_\ell, x_0)$ and find $\hat{u}(x)$ to obtain a pH system

$$\begin{bmatrix} \dot{x}_\ell \\ \dot{x}_0 \end{bmatrix} = \begin{bmatrix} J_\ell(x) - R_\ell(x) & 0 \\ 0 & J_0(x) - R_0(x) \end{bmatrix} \nabla H(x),$$

where $H(x) := H_\ell(x_\ell) + H_0(x_0)$ and $R_0(x) = R_0^\top(x) \geq 0$

- Control objectives: Ensure $\lim_{t\to\infty} x(t) = 0$ or

$$\lim_{t\to\infty} H_\ell(x_\ell(t)) = \beta_\ell > 0, \quad \lim_{t\to\infty} x_0(t) = 0,$$

Assumptions for EPD IDA-PBC

(C1) (Matching PDE)

$$A_\ell^\top(x)\Big[J_\ell(x) - R_\ell(x)\Big]\nabla H_\ell(x_\ell) + A_0^\top(x)\Big[J_0(x) - R_0(x)\Big]\nabla H_0(x_0) = 0.$$

(C2) (EPD condition) $[R_\ell(x) + R_\ell^\top(x)](H_\ell(x_\ell) - \beta_\ell) \geq 0$.

(C3) (Stabilization condition) $\arg\min_{x\in\mathbb{R}^n} H(x) = 0$, (isolated).

(C4) (Detectability) Define the function

$$Q(x) := \|\nabla H_0(x_0)\|^2_{R_0(x)} + \frac{1}{2}(H_\ell(x_\ell) - \beta_\ell)\|\nabla H_\ell(x_\ell)\|^2_{R_\ell(x)+R_\ell^\top(x)}.$$

For the closed-loop system, there exists $h(x) \in \mathbb{R}$ such that

$$\Big[Q(x(t)) \equiv 0 \text{ and } x(0) \notin \mathcal{I} := \{x \in \mathbb{R}^n \mid h(x) = 0\}\Big] \Rightarrow \text{control objective}$$

Notation: $x \in \mathbb{R}^n$, $P \in \mathbb{R}^{n\times n}$ and $P > 0$: $\|x\|_P := x^\top P x$.

Main Result

Nonholonomic system (NHS) verifying (C1)-(C4), in closed-loop with $u = \hat{u}(x)$

$$\hat{u}(x) := [S^\top(x)S(x)]^{-1}S^\top(x)\begin{bmatrix}\Big(J_\ell(x) - R_\ell(x)\Big)\nabla H_\ell(x_\ell) \\ \Big(J_0(x) - R_0(x)\Big)\nabla H_0(x_0)\end{bmatrix}.$$

For all initial conditions outside $\mathcal{I}$:

- If $\beta_\ell > 0$

$$\lim_{t\to\infty} H_\ell(x_\ell(t)) = \beta_\ell, \quad \lim_{t\to\infty} x_0(t) = 0,$$

- If $\beta_\ell = 0$,

$$\lim_{t\to\infty} x(t) = 0,$$

Remarks

- Does not contradict Brockett's necessary condition, we prove the origin is Lyapunov stable but not asymptotically stable. We prove

$$\Big[\ |x(0)| < \delta,\ x(0) \notin \mathcal{I}\ \Big] \ \Rightarrow\ \lim_{t\to\infty} x(t) = 0,$$

- Assumption $x = \mathrm{col}(x_\ell, x_0)$ relaxed to

$$\begin{bmatrix} x_\ell \\ x_0 \end{bmatrix} = \mathcal{P}x,$$

P where $\mathcal{P} \in \mathbb{R}^{n\times n}$ is a permutation matrix.

- If $\beta_\ell > 0$ the origin $x = 0$ is an unstable equilibrium

Application to the Nonholonomic Integrator

- Dynamics $\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \dot{x}_3 = x_2 u_1$.
- Conditions (C1)-(C4) satisfied with

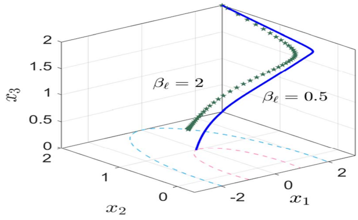
$$
\begin{aligned}
H_\ell(x_1, x_2) &= \frac{1}{2}(x_1^2 + x_2^2),\ H_0(x_3) = \frac{1}{2}x_3^2 \\
J_0 = 0,\ J_\ell(x_3) &= \begin{bmatrix} 0 & -x_3 \\ x_3 & 0 \end{bmatrix},\ R_0(x_2) = x_2^2 \\
R_\ell(x_1, x_2) &= \begin{bmatrix} 0 & 0 \\ 0 & \gamma[H_\ell(x_1, x_2) - \beta_\ell] \end{bmatrix},\ \gamma > 0 \\
Q(x) &= -x_2^2\left[\gamma(x_1^2 + x_2^2 - \beta_\ell)^2 + x_3^2\right] \\
h(x) &= (x_1^2 + x_2^2)(x_2^2 + x_3^2).
\end{aligned}
$$

- Smooth, time-invariant control law

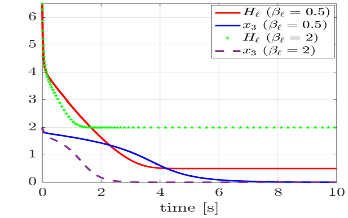
$$
\hat{u}(x) = \begin{bmatrix} -x_2 x_3 \\ x_1 x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -\gamma[H_\ell(x_1, x_2) - \beta_\ell]x_2 \end{bmatrix}.
$$

Comparative Simulations vs Pomet's and Astolfi's

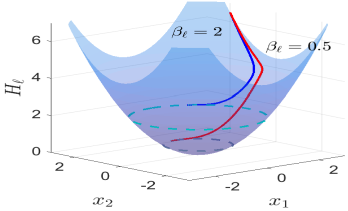
Energy Regulation $\gamma = 5$



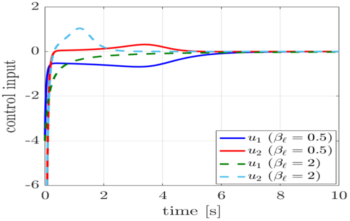
(a) Trajectories in the state space



(b) $x_3(t)$ and $H_\ell(x_\ell(t))$

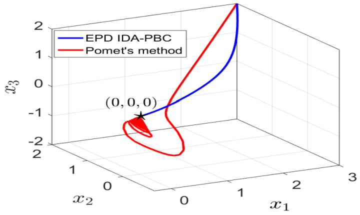


(c) Trajectories of x_ℓ and $H_\ell(x_\ell)$

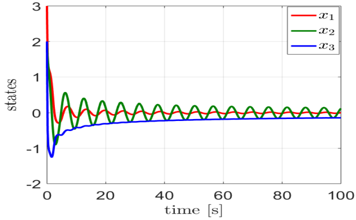


(d) Control inputs

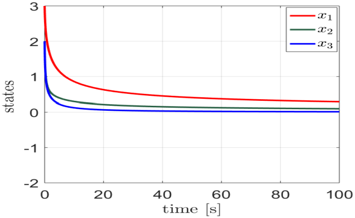
State Regulation and Pomet's Method



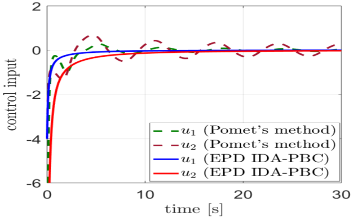
(e) Trajectories in the state space



(f) $x(t)$ of Pomet's controller

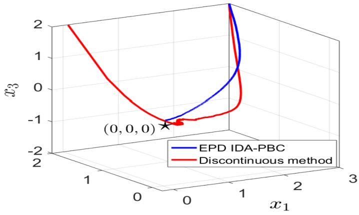


(g) $x(t)$ of EPD IDA-PBC

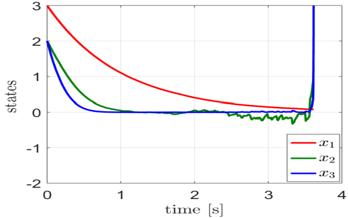


(h) Control inputs

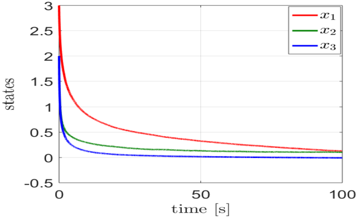
State Regulation and Astolfi's Method (with Noise)



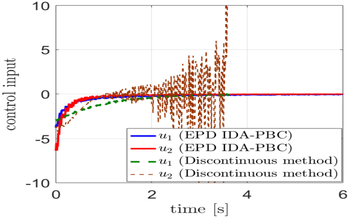
(i) Trajectories in the state space



(j) $x(t)$ of Astolfi's controller



(k) $x(t)$ of EPD IDA-PBC



(l) Control inputs

Path Following of Underactuated Mechanical Systems

Problem Formulation

- port-Hamiltonian mechanical systems

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_{n\times n} & A(q) \\ -A^\top(q) & -R(x) \end{bmatrix} \nabla H(x) + \begin{bmatrix} 0_{n\times m} \\ G(q) \end{bmatrix} u \tag{2}$$
$$q_y = h(q).$$

- A desired smooth path characterized by the Jordan curve $\Phi(q_y) = 0$, design a controller that achieves:

P1 *boundedness*: the states x are bounded for all initial conditions;

P2 *convergence and invariance*: the states x will converge to the path set

$$\lim_{t\to\infty} \|x(t)\|_{\mathcal{C}_x} = 0,$$

where we defined

$$\mathcal{C}_x := \{x \in \mathbb{R}^{2n} \mid \Phi(h(q)) = 0\}. \tag{3}$$

which is an invariant set;

P3 *forward motion*: $\dot{x} \neq 0$ for all $t \geq 0$ and $x \in \mathcal{C}_x$.

Conventional Methods

- Reference points/guidance laws: the desired path $\mathcal{C}$ is parameterized as $q_y^d = \sigma(\theta)$, with $\theta \in [0, L]$ and a mapping $\sigma : [0, L] \to \mathbb{R}^2$. The variable θ propagates according to the motion generator dynamics

$$\dot{\theta} = \Gamma(\theta, x), \quad \theta(0) = \theta_0. \tag{4}$$

  (multiple projection, no guarantee of invariance)
- Set stabilization: transverse feedback linearization + virtual holonomic constraints (VHC)
  (online optimization, local results)
- Orbital stabilization provides a new problem formulation (or a constructive framework).

Immersion and Invariance Path Following

- New design: I&I orbital stabilization + MSEA
- Use a time-varying target oscillator to shape the behaviour (parameterized by systems state x)
- Target oscillator

$$\dot{\xi} = \begin{bmatrix} -\mathcal{R}_1(\xi) & \frac{a(\xi,x)}{\Phi(\xi)} \\ -\frac{a(\xi,x)}{\Phi(\xi)} & -\mathcal{R}_2(\xi) \end{bmatrix} \nabla V_d$$

  with

$$V_d(\xi) = \frac{1}{2}|\Phi(\xi)|^2$$

- Almost global convergence
- Be able to deal with underactuated mechanical systems
- By adding a suitable integral action to the target oscillator, we are able to attenuate the unmatched external disturbances.

Summary

C1. Oscillation Generation via

- Immersion and Invariance (I&I)
  (R. Ortega, B. Yi, J. Romero and A. Astolfi, *Int. J. of Robust and Nonlinear Control*'19)
- Interconnection and Damping Assignment (IDA) Passivity-based Control (PBC)
  (B. Yi, R. Ortega and D. Wu, *Automatica'19)*
- Mexican Sombrero Energy Assignment (MSEA)
- Energy Pumping and Damping (EPD)

C2. Application to Mechanical Systems

- Smooth, Time-invariant Regulation of Nonholonomic Systems
  (B. Yi and R. Ortega, *Int. J. of Robust and Nonlinear Control*'19, under review)
- Path Following of Underactuated Mechanical Systems

Signal Injection of EM Systems

C1 Second-Order Averaging Analysis of Signal Injection to MIMO Systems

(B. Yi, R. Ortega, *Systems & Control Letters*'18)

C2 Linear Regressor Perspective

(B. Yi, et al., *European J. Control*'19)

- New Virtual Output Estimator based on Dynamic Regressor Extension and Mixing (DREM)
- Electrical Coordinate Observers for General Electromechanical Systems

C3 Frequency Domain Perspective

(B. Yi, et al., *IET Power Electronics*'19, under review)

- Frequency Domain Analysis to Conventional Methods (first quantitative analysis, IEEE CSS BSPA)
- Connection between Conventional Methods and New Design

Nonlinear Observer Theory

C1 [KKL+PEB] Observer: A New Design and A Unifying Framwork for General Nonlinear Systems

(B. Yi, R. Ortega, *IEEE Trans Automatic Control*'19)

- A New Observer Design Technique—[KKL+PEB] Observer—Combining KKLO and PEBO in a Seamless Way
- Recast as Particular Cases of Immersion & Invariance Observers
- Some Physical Examples

C2 Constructive Nonlinear Observers

(B. Yi, R. Ortega, *Systems & Control Letters*'18)

- Relaxing the Assumptions in Parameter Estimation-Based Observers
- An Approach to Solve the PDE
- Application to Mechanical Systems

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THANK YOU

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