# A Tutorial on Parameter Estimation-Based Observers

## Theory and Robotic Applications

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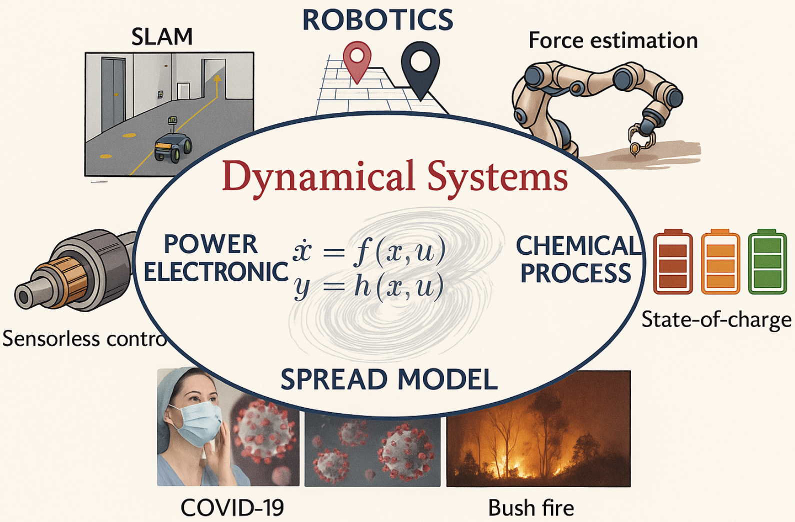
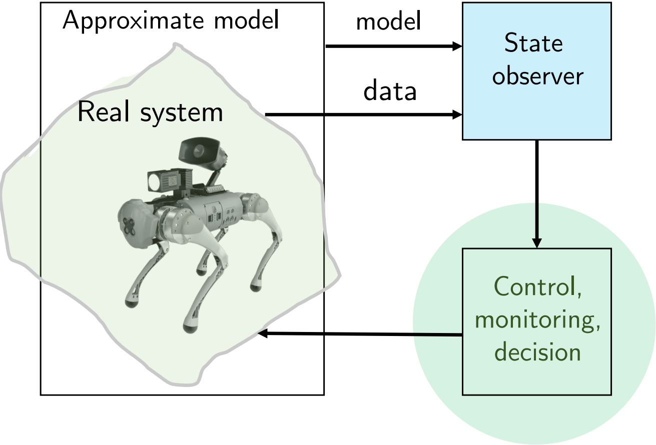
# Outline

1. Background and Motivations
2. Part I: Parameter Estimation-Based Observer on $\mathbb{R}^n$
3. Part II: PEBO on Manifolds & Robotic Estimation
4. From PEBO to Gradient Observers
5. Summary

# Hidden Variables



# Estimation as the Heart of Robotic Systems



# Problem Formulation: Observer Design

- Given measurements (i.e. partial/posterior information), try to estimate hidden variables of a **dynamical system**.


$$\dot{x} = f(x, u), \quad y = h(x, u) \tag{1}$$

Problem Set: Design an observer dynamics

$$\dot{\xi} = H(\xi, y, u), \quad \hat{x} = N(\xi, y, u)$$

such that

$$\lim_{t \to +\infty} |\hat{x}(t) - x(t)| = 0.$$

# Key Challenges: A Personal Perspective

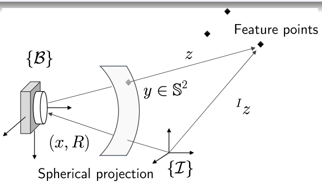
1. Not dual to nonlinear control, some general tools:
   - Extended Kalman filter (locality)
   - High-gain observer (sensitive to noise)
   - Kazantzis-Kravaris-Luenberger (KKL) observer (inverse mapping)

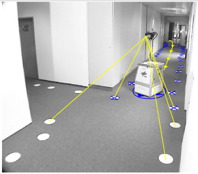
$$\dot{x} = f(x),\ y = h(x) \ \xrightarrow[z=T(x)]{} \ \dot{z} = Az + B(y)$$

$$T(x) = \int_{-\infty}^{0} \exp(-As)B_{1m}b(h\breve{X}(x,s)))ds$$

2. Weakly observable systems

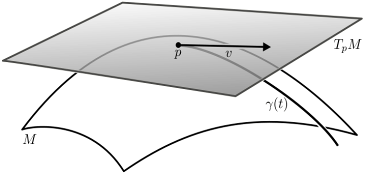
> These observers require uniform observability. In robotics, we need to estimate systems along specific trajectories.



# Key Challenges: A Personal Perspective

4. Not satisfy the observable canonical forms
5. State constraints
   - In many robotic estimation problems, states live on manifolds.




6. Adaptive observers
   - Chicken or the egg: Joint estimation of state and parameters – not dual to adaptive control

$$\dot{x} = f(\theta, x, u), \quad y = h(\theta, x)$$

# Outline

# Fundamental Idea of PEBO

Facts:

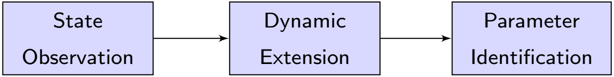
- Nonlinear observability relies on system trajectories.
- Hard to verify in advance: Persistency of excitation (PE), uniform complete observability (UCO)
- Parameter estimation is simpler than state identification.

Idea: Translate state observation into online parameter identification.

(1) Simplify state observation

(2) Fundamental challenge of weak observability ("non-informative")

(3) Nonlinearity in robotics, mechanical, and power systems



# A Motivating Example

**Example (An LTI system)**

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ u\end{bmatrix} \qquad (2)$$

$$y = x_1$$

- Available derivative of unknown $x_2$. Design a dynamic extension

$$\dot{z} = u$$

- Constant error $\frac{d}{dt}(z - x_2) = 0 \implies x_2 = z + \theta$ [1]
- Estimating $x_2 \Leftrightarrow$ estimating $\theta$
- Linear regressor

$$\dot{y} - z = \theta \quad (\dot{y}?)$$

- Filtering

$$\frac{\lambda s}{\lambda + s}[y] - \frac{\lambda}{\lambda + s}[z] = \theta + \epsilon_t$$

[1] $\theta := x_2(0) - z(0)$

# Parameter Estimation-Based Observer (PEBO)[2]

How to translate state observation into parameter identification?

1. Transformability (Feasibility to get a regression model)

- Coordinate transformation: $x \mapsto z := \phi(x)$ s.t. $\dot{z} = H(y, u)$
- Left invertibility: $\phi^{\mathrm{L}} \circ \phi = \mathrm{I_d}$

  - Dynamic extension: $\dot{\xi} = H(y, u)$
  - Re-parameterization: $z(t) = \xi(t) + \theta$
  - Regression model: $$\boxed{y = h(\phi^{\mathrm{L}}(\xi + \theta), u)} \qquad (\dagger)$$

2. Identifiability

- Is $\theta$ identifiable in ($\dagger$)? Related to rank $\{\frac{\partial y}{\partial \theta}\}$
- Algorithm

[2] R. Ortega *et al.*, SCL'15, A parameter estimation approach to state observation of nonlinear systems.

# Transformability in PEBO

Transform $\dot{x} = f(x,u),\ y = h(x)$ into $\dot{z} = H(y,u)$

- PDE Solvability (Search for $\phi$ and $H$)

$$\frac{\partial \phi}{\partial x}(x) f(x,u) = H(h(x),u). \tag{3}$$

- Left invertibility (Search for $\phi^{\mathtt{L}}$)

$$\phi^{\mathtt{L}}(\phi(x)) = x, \quad \forall x. \tag{4}$$

Connected to the PDE in Kazantzis-Kravaris-Luenberger (KKL) observer[3]

$$\frac{\partial \phi}{\partial x}(x) f(x,u) = A\phi(x) + H(h(x),u) \tag{5}$$

- KKL observer: Hurwitz matrix $A$;
- PEBO: $A = 0_{n\times n}$. [Extensible to Re$\{\lambda(A)\} \le 0$]

[3] Yi, Ortega & Zhang, IEEE TAC'19, On state observers for nonlinear systems: A new design and a unifying framework.

# Transformability in PEBO (cont'd)

A long list of physical models satisfy the PDE:

1. Electromechanical (sensorless)

$$\underbrace{\dot{\lambda}}_{\text{flux}} = -R \underbrace{\mathbf{i}}_{\text{current}} + \underbrace{\mathbf{v}}_{\text{voltage}} \tag{6}$$

2. Mechanical (partially linearizable via CC)

$$\underbrace{\dot{p}}_{\text{momentum}} = -\mathcal{T}^{\top}(q)\left[\nabla U(q) - G(q)u\right] \tag{7}$$

3. Chemical Process

$$\underbrace{\dot{x}}_{\text{concentration}} = \underbrace{\mu(s)x}_{\text{flow rate } y} \tag{8}$$

4. Robotics (Introduce later ...)
   Depth estimation, navigation, SLAM, IMU preintegration

# Parameter Identification

State Observation → Dynamic Extension → Parameter Identification

- Have obtained the regression model

$$y = \phi_h(\theta, t) := h(\phi^{\mathrm{L}}(\xi + \theta), u) \tag{9}$$

- Once obtaining $\hat{\theta}$, the state can be reconstructed as

$$\hat{x} = \phi^{\mathrm{L}}(\xi + \hat{\theta}) \tag{10}$$

- Identification can be done by
  - Solving (9) at each instance ("offline" optimization)
  - Online optimization: flows of dynamical systems, recursive algorithms, robust
- Observability: becomes the identifiability of the regressor (9)

We are able to design observers under weak observability – NOT uniformly over time.

# Remarks on PEBO

- Weak excitation: (Wang *et al.*, IJC'23)
  - Linear regressor:

$$\text{Identifiability} \iff \text{IE} \qquad (11)$$

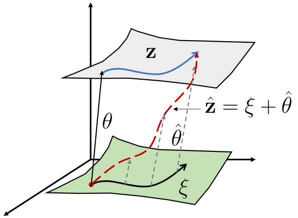
Interval excitation (IE): $\exists\ \gamma, t_c > 0$

$$\int_0^{t_c} \phi(s)\phi(s)^\top ds \succeq \gamma I_n \qquad (12)$$

  - DREM (dynamic regressor extension and mixing) estimator: GES under IE (Ortega *et al.*, SCL'22), (Bobtsov *et al.*, TAC'22), (Yi *et al.*, CDC'21)

- Geometric interpretation: PEBO generates an invariant foliation (Yi et al., TAC'19)

$$\{(x,\xi) : \xi = \phi(x) + \theta\}.$$




Identify the leaf of the foliation where the system evolves.

# Generalized (G)PEBO: Basic Idea [4]

- Relaxed Transformability
  - PEBO:

$$\dot{z} = B(y,u)$$

  - GPEBO:

$$\dot{z} = A(y,u)z + B(y,u)$$

  PDE solvability

$$\frac{\partial \phi}{\partial x}(x) f(x,u) = A(y,u)\phi(x) + B(y,u). \tag{13}$$

- Dynamic extension

$$\dot{\xi} = A(y,u)\xi + B(y,u), \quad \xi(0) = \xi_0 \tag{14}$$

[4] Ortega *et al.*, Automatica'21, Generalized parameter estimation-based observers: Application to power systems and chemical–biological reactors.

# Generalized (G)PEBO

- Re-parameterization

$$\dot{\overbrace{z-\xi}} \;=\; A(y,u)(z-\xi)$$

$$\Downarrow$$

$$\boxed{z(t) \;=\; \xi(t) + \Phi(t,0)[\theta - \xi_0]}$$

where $\Phi(t,s)$ is the state transition matrix:

$$\Phi(t,s) = \Omega(t)\Omega(s)^{-1}$$

and the fundamental matrix $\Omega$ of $A(y,u)$.

# GPEBO: Implementation

- Model: $(A_t, B_t, C_t)$

$$\dot{x} = A(y,u)x + B(y,u), \quad y = C(y,u)x$$

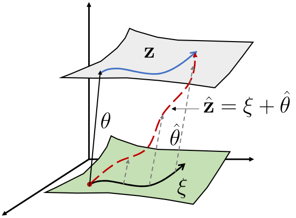
- Observer:
  - Dynamic extension

$$\dot{\xi} = A_t\xi + B_t, \quad \xi(0) = \xi_0$$
$$\dot{\Omega} = A_t\Omega, \quad \Omega(0) = I_n$$

  - STM: $\Phi(s,t) = \Omega(t)\Omega(s)^{-1}$
  - Algebraic equation:

$$x_t = \xi_t - \Omega_t\xi_0 + \Omega_t\theta \qquad (15)$$

  - Online optimization $\theta$: gradient, LS




Linear Regression on $\theta$

$$Y_t = C_t\Omega_t\theta$$

# Remarks

1. Convergence condition: $\exists\ t_1 > t_0 \geq 0$

$$W(t_0, t_1) := \int_{t_0}^{t_1} \Phi^\top s, t_0 C^\top(s) C(s) \Phi(s, t_0) ds \succeq \gamma I_n. \quad (16)$$

   Weaker than uniform complete observability.

2. Boundedness for unstable $A(\cdot)$:

$$\begin{aligned} \dot{\hat{x}} &= A(t)\hat{x} + FC(t)^\top[y - C(t)\hat{x}] \\ \dot{F} &= A(t)F - FC^\top(t)C(t)F + ? \end{aligned} \quad (17)$$

   to generate invariant foliation

$$\{F^{-1}(\hat{x}(t) - x(t)) = \text{constant}\}.$$

3. Finite-time convergent observer
   - Estimation of constant parameters is easier
   - Different from the existing finite-time observer techniques

4. Adaptive observer: Joint estimation of parameters and states

# Finite-Time Observer: Reduced-Order Case[5]

Consider a specific coordinate

$$\begin{aligned} \dot{x} &= A_x(y,u)x + b_x(y,u) \\ \dot{y} &= A_y(y,u)x + b_y(y,u). \end{aligned} \tag{18}$$

Derivation of linear regression model:

- Dynamic Extension:

$$\begin{aligned} \dot{\xi}_x &= A_x(y,u)\xi_x + b_x(y,u) \\ \dot{\Omega}_x &= A_x(y,u)\Omega_x, \quad \Omega_x(0) = I_{n_x}. \end{aligned} \tag{19}$$

- Re-parameterization: $x(t) = \xi_y(t) + \Omega_x(t)\theta$
- Regression model: Substitute to the dynamics of $y$

$$\dot{y} = A_x(y,u)[\xi_y + \Omega\theta] + b_y(y,u) \tag{20}$$

- Feasible realisation: Filtering $\implies$ $q_y = m_y^\top \theta$

$$\underbrace{\frac{\lambda p}{\lambda + p}[y] - \frac{\lambda}{\lambda + p}[A_y(y,u)\xi_y + b_y(y,u)]}_{q_y} = \underbrace{\frac{\lambda}{\lambda + p}[A_y(y,u)\Omega]}_{m_y^\top}\theta \tag{21}$$

[5] Bobtsov, Ortega, Yi & Nikolaev, IJC'22, Adaptive state estimation of state-affine

## Proposition (Reduced-order finite-time observer)

For the system and the above dynamic extension, the DREM estimator

$$
\begin{aligned}
\dot{r} &= -\lambda r + \lambda m_y q_y, \quad r(0) = 0_{n_x \times n_y} \\
\dot{Q} &= -\lambda Q + \lambda m_y m_y^\top, \quad Q(0) = 0_{n_x \times n_x} \\
\dot{\hat{\theta}} &= -\gamma \Delta[\Delta \hat{\theta} - Y] \\
\dot{\omega} &= -\gamma \Delta^2 \omega, \quad \omega(0) = 1
\end{aligned}
$$

- Dynamic extension for full use of "historical information"
- Mixing to get scalar regressors.

with the gain $\gamma > 1, \rho \in (0,1)$ and the mappings

$$\Delta := \det\{\Omega\}, \quad Y := \mathrm{adj}\{\Omega_x\} r, \quad \omega_c = \begin{cases} \omega & \text{if } \omega \in [0, \rho) \\ \rho & \text{if } \omega \in [\rho, 1] \end{cases} \tag{22}$$

and the estimate

$$\hat{x} = \xi + \frac{1}{1-\omega_c} \Omega_x [\hat{\theta} - \omega_c \hat{\theta}(0)] \tag{23}$$

achieves finite-time convergence if $\Delta(t)$ is intervally excited (IE).

Mazenc's design:

$$\begin{aligned}\hat{x}(t) =& \kappa(T)^{-1}[L(t)y(t-T) - \Phi_H(T,0)L(t)y(t)] \\ &+ \Phi_H(T,0)\gamma_1(t) - \gamma_1(t-T) \\ &- \kappa(T)^{-1}[\Phi_{A_x}(T,0)\gamma_2(t) - \gamma_2(t-T)]\end{aligned}$$

- Assumption (PE-like): $\exists \tau > 0$ and bounded $L(t)$ such that $H(t) = A_x(t) + L(t)A_y(t)$ with (i)

$$\kappa(t) = \Phi_H(t, t-\tau) - \Phi_{A_x}(t, t-\tau)$$

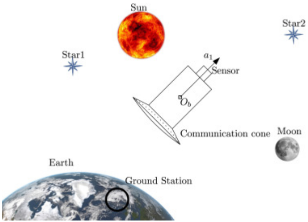
  is invertible for all $t \in \mathbb{R}$ and (ii) the inverse $\kappa^{-1}(t)$ is bounded.

One motivation for finite-time observers is to solve the estimation problems that do not satisfy sufficient excitation.

# Outline

# PEBO on Matrix Lie Groups: Motivation

- Attitude of a rigid body: **orientation** w.r.t. an inertial reference frame.
- Applications: navigation and localization (robotics & aerospace)
- Unmeasurable




Attitude dynamics:

$$\dot{R} = R\omega_\times \tag{24}$$

and output (depending on sensors)

$$y = h(R, t)$$

| | |
|---|---|
| $R \in \mathsf{SO}(3)$ | Rotation matrix {B} relative to {I} |
| $\omega \in \mathbb{R}^3$ | Rotational velocity in {B} |
| $\omega_\times \in \mathsf{so}(3)$ | $\omega_\times : \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$ |

State Observation → **Dynamic Extension** → Parameter Identification

## Transformability:

- Error definition:

$$\boxed{R_1R_2^\top \text{ rather than Euclidean error } x_1 - x_2}$$

- Dynamic extension:

$$\dot{Q} = Q\omega_\times, \quad Q_0 = I_3$$

- Re-parameterization:

$$\dot{\overbrace{RQ^\top}} = \dot{R}Q^\top - RQ^\top\dot{Q}Q^\top = 0, \tag{25}$$

we have for all $t, s \geq 0$

$$R(t)Q(t)^\top = R(0)Q(0)^\top \iff \boxed{R(t) = R_0Q(t)} \tag{26}$$

- Output becomes the regression model

$$y = h(R_0 Q, t),$$

  with constant $R_0 \in \mathrm{SO}(3)$
- Hold true on other matrix Lie groups, e.g. SE(3)
- $E$-synchronous of $Q, R$ (Lageman, Trumpf & Mahony, TAC'09)
- Effective in solving various robotic estimation problems

# Necessary and Sufficient Condition for Attitude Estimation[6]

## Model

- Dynamics:

$$\dot{R} = R\omega_\times, \quad R \in SO(3),$$

- Outputs: Multiple vector measurements $(n_1, n_2 \in \mathbb{N})$

1. Complementary: known inertial vectors, measured in {B}

$$y_{\mathtt{B},i} \ = \ R^\top g_i, \quad i \in \ell_1 := \{1, \ldots, n_1\} \tag{27}$$

[magnetic field, gravitational force]

2. Compatible: known vectors in the body frame, measured in {I}

$$y_{\mathtt{I},j} \ = \ R b_j, \quad j \in \ell_2 := \{1, \ldots, n_2\} \tag{28}$$

[two GPS receivers attached to the rigid body with a base-line]

[6] Yi, Wang & Manchester, TAC'23 Attitude estimation from vector measurements: Necessary and sufficient conditions and convergent observer design.

# Problem Sets

P1 Observability: An **iff** condition to distinguishability;
P2 Observer design

**Definition (Distinguishability)**

Consider an open set $\mathcal{X} \subset \mathbb{R}^n$ and a complete system

$$\dot{x} = f(x, t), \quad y = h(x, t). \tag{29}$$

The system (29) is distinguishable on $\mathcal{X}$ if for all $(x_a, x_b) \in \mathcal{X} \times \mathcal{X}$,

$$h\big(X(t; t_0, x_a), t\big) = h\big(X(t; t_0, x_b), t\big), \ \forall t \geq t_0 \implies x_a = x_b.$$

# Necessary and Sufficient Condition to Observability

## Proposition (Observability)

The attitude model with $n := n_1 + n_2 \geq 1$ is distinguishable if and only if there exist two moments $t_1, t_2 \geq 0$ such that

$$\sum_{i,l\in\ell_1,j,k\in\ell_2} \left|g_i(t_1)_\times g_l(t_2)\right| + \left|g_i(t_1)_\times R_0\Phi(0,t_2)b_j(t_2)\right| + \left|b_j(t_1)_\times \Phi(t_1,t_2)b_k(t_2)\right| > 0, \tag{30}$$

in which $\Phi(t,s)$ is the state transition matrix of the time-varying system matrix $-\omega_\times(t)$ from $s$ to $t$.

# Sketch of Proof

- PEBO on SO(3): The state transition matrix $\Phi(t,s)$ is defined as

$$\frac{\partial}{\partial t}\Phi(t,s) \;=\; -\omega_\times(t)\Phi(t,s), \quad \Phi(s,s) \;=\; I_3. \tag{31}$$

- Reparameterization: $R(t) \;=\; R_0\Phi(0,t)$
- Regression Model on the initial condition $R_0$:

$$\boxed{Y(t) = R_0^\top \phi(t),} \quad R_0 \in SO(3) \tag{32}$$

with

$$Y = \Phi(0,t)\left[y_{\mathtt{B},1},\ldots,y_{\mathtt{B},n_1},b_1,\ldots,b_{n_2}\right], \phi = \left[g_1,\ldots,g_{n_1},y_{\mathtt{I},1},\ldots,y_{\mathtt{I},n_2}\right].$$

- A Wahba problem with *infinite* numbers of eqs *over time*.
- Solvability $\iff$ $\exists t_1,t_2 \geq 0$ and $\exists i,j \in \{1,\ldots,n\}$ s.t.

$$\boxed{\phi_i(t_1)_\times \phi_j(t_2) \neq 0} \tag{33}$$

- Rewritten as three cases.

# Attitude Observer Design (Minimal Condition)

- A single complementary measurement case – more challenging
- Model:

$$\dot{R} = R\omega_{\times}, \quad y_{\mathtt{B}} = R^{\top} g(t). \tag{34}$$

- The necessary and sufficient condition becomes

$$\exists t_1, t_2 > 0, \quad \left| g(t_1)_{\times} g(t_2) \right| > 0. \tag{35}$$

- Key Idea: Integral error term providing historical information
- PEBO design
  - Dynamic extension: $\dot{Q} = Q\omega_{\times}$.
  - Online parameter estimation: $\dot{\hat{Q}}_c = \eta_{\times} \hat{Q}_c$
  - Observer output: $\hat{R} = \hat{Q}_c^{\top} Q$
  - Functions:

$$\begin{aligned} &\eta = \gamma_{\mathtt{P}} (\hat{Q}_c g)_{\times} (Q y_{\mathtt{B}}) + \gamma_{\mathtt{I}} \xi, \quad \xi = 2\mathsf{vex}\big(\mathsf{skew}(A \hat{Q}_c^{\top})\big) \\ &\dot{A} = \begin{cases} Q y_{\mathtt{B}} g^{\top}, & t \in [0, T) \\ 0_{3\times 3}, & t \geq T \end{cases} \quad \text{with} \quad \mathsf{vec}(\omega_{\times}) = \omega \end{aligned} \tag{36}$$

- Almost global asymptotic stability

# Discussions

- The error term $\eta$ contains two parts

$$\eta = \underbrace{\gamma_{\mathrm{P}}(\hat{Q}_c g)_\times (Qy)}_{\text{current}} + \underbrace{\gamma_{\mathrm{I}}\xi}_{\text{historical}} . \tag{37}$$

  A "proportional + integral"-type error term.

- (Trumpf *et al.*, IEEE TAC 2012) a *sufficient* (not necessary) condition

$$\lambda_2\left(\sum_{i\in\ell_1}\int_0^T g_i(s)g_i^\top(s)ds\right) + \left\|\int_0^T \sum_{j\in\ell_2}\left(\omega_\times b_j(s) + \frac{d}{ds}b_j(s).\right)ds\right\| > 0 \tag{38}$$

$$\boxed{(38) \quad \Longrightarrow \quad \text{Proposed necessary and sufficient condition}}$$

- Trivially extended to delayed, intermittent and biased measurements.
- Another solution by introducing a virtual reference vector.

# Simulations

A single time-varying inertial vector $g(t) = \begin{cases} e_1, & t \in [0, 5)\text{s} \\ e_3, & t \geq 5\text{s}, \end{cases}$

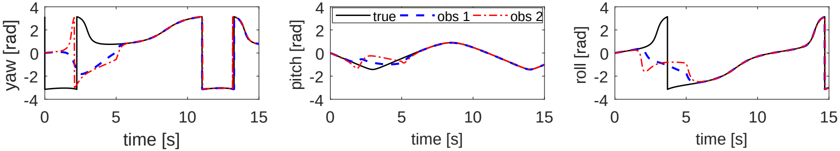



Figure 2: Performance of the attitude observer with Euler angles

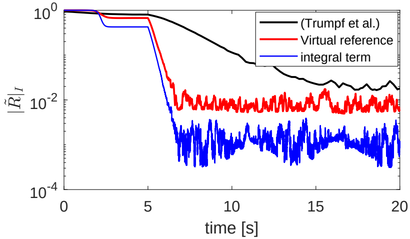



Figure 3: Comparison of the norms of estimation errors $|\tilde{R}|_I$

# With Gyro Bias (Constant Reference) [7]

**Gyro bias**

$$\dot{R} = R\omega'_\times$$
$$y = R^\top g$$
$$\omega = \omega' + \omega_b$$

IMU provides the rotational velocity $\omega$ with an unknown bias $\omega_b$.

For constant reference vector $g$, we have

$$\dot{y} = -\omega_\times y - y_\times \omega_b$$

$$\Longrightarrow \quad \underbrace{\frac{\alpha p}{\alpha + p}[y] + \frac{\alpha}{\alpha + p}[\omega_\times y]}_{Y} = \underbrace{-\frac{\alpha}{\alpha + p}[y_\times]}_{\phi^\top(t)} \omega_b$$

Estimate $\omega_b$ and then cascade to the attitude observer.

[7] Yi et al., IFAC NOLCOS'2022, Online gyro bias estimation from single vector measurements using regression models.

# With Gyro Bias (Time-varying Reference)

- For time-varying reference vectors

$$\dot{g} = (\Lambda_g)_\times g,$$

we have

$$\dot{y} - (\omega - \omega_b)_\times y = R^\top (\Lambda_g)_\times g$$

- Using $R \in SO(3)$,

$$h(\omega_b, t) := |\dot{y} + (\omega - \omega_b)_\times y|^2 - |\Lambda_g \times g|^2 = 0$$

and the nonlinear regressor

$$H(\theta, t) := Y(t) + \Phi(t)\theta + \theta^\top D(t)\theta = 0.$$

**Convexified gradient observer**

$$\dot{\hat{\omega}}_b = -\gamma \text{max}\{0, H(\hat{\omega}_b, t)\} \cdot \nabla_{\omega_b} H(\hat{\omega}_b, t)$$

Under some technical conditions on $g(t)$, we may achieve global convergence $|\hat{\omega}_b(t) - \omega_b| \to 0$ as $t \to \infty$.

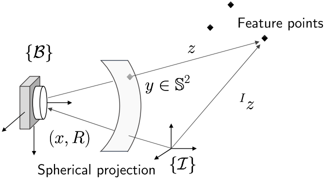
# Range/Depth Estimation for Visual Features [8]

## Spherical Projection

- The output from a single *monocular* camera is the bearing

$$y = \frac{z}{|z|} \in \mathbb{S}^2.$$

- Range estimation: reconstruct $|z|$
- Applications:
  1. Structure from motion (SfM)
  2. Bearing-only formation control
  3. Image-based visual servoing
  4. Visual inertial SLAM
  5. ...




[8] Yi, Jin & Manchester, Automatica'22, Globally convergent visual-feature range estimation with biased inertial measurements.

# Motion Model

- A constant feature point ${}^{I}z$ in {I}, and its coordinate in {B} is

$$z = R^{\top}({}^{I}z - x).$$

- The dynamics is

$$\begin{aligned} \dot{x} &= Rv, \quad x \in \mathbb{R}^3 \\ \dot{R} &= R\Omega_{\times}, \quad R \in SO(3) \\ \dot{v} &= -\Omega_{\times} v + a + b_a + \boxed{R^{\top} g}. \end{aligned} \tag{39}$$

- Inertial Measurements: acceleration $a$, velocity $\omega$, and bearing $y = \frac{z}{|z|}$

Challenges:

1. No results available with only inertial measurements.
   How can we deal with the unknown attitude $R \in \mathrm{SO}(3)$?
2. Uniform observability is strong – hardly satisfied in robotics.

# Solution to address $R^\top g$

- Design a dynamic extension

$$\dot{Q} = Q\Omega_\times$$

- Existence of a constant matrix $Q_c \in SO(3)$ satisfying

$$R(t) = Q_c Q(t), \quad \forall t \geq 0. \tag{40}$$

- The last term in $\dot{v}$ is re-parameterised as

$$\boxed{R(t)^\top g = Q(t)^\top g_c,} \quad \forall t \geq 0$$

with a new *constant unknown* vector $g_c \in \mathbb{R}^3$ defined as $g_c := Q_c^\top g$.

| |
|---|
| Unknown $R(t)$ and $g$ $\longrightarrow$ Unknown constant $g_c$ |

# Generation of Linear Regression Models

- Defining the extended state $\chi := \text{col}(r, v, b_a, g_c) \in \mathbb{R}^{10}$, thus

$$\dot{\chi} = A(y, \Omega, Q)\chi + B(a) \tag{41}$$

with

$$A := \begin{bmatrix} 0 & -y^\top & 0_3^\top & 0_3^\top \\ 0_3 & -\Omega_\times & I_3 & Q^\top \\ 0_6 & \dots & \dots & 0_{6\times 3} \end{bmatrix} \quad B := \begin{bmatrix} 0 \\ a \\ 0_6 \end{bmatrix}. \tag{42}$$

- PEBO leads to the algebraic relation

$$\chi = \xi + \Psi\theta, \quad \text{col}(z, v, b_a) = \begin{bmatrix} y \\ & [I_6 \quad 0_{6\times 3}] \end{bmatrix} [\xi + \Psi\theta].$$

- Using the dynamics

$$\dot{y} = -\Omega_\times y - \frac{1}{|z|}\Pi_y v \quad \Longleftrightarrow \quad r(\dot{y} + \Omega_\times y) = -\Pi_y v$$

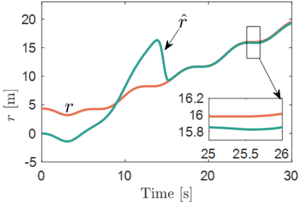
- Unavailability of $\dot{y}$: Filter $\frac{\alpha}{p+\alpha}$ and apply the swapping lemma

$$\frac{\alpha}{p+\alpha}[xy] = y\frac{\alpha}{p+\alpha}[x] - \frac{1}{p+\alpha}\left[\dot{y}\frac{\alpha}{p+\alpha}[x]\right], \quad p := \frac{d}{dt} \tag{43}$$

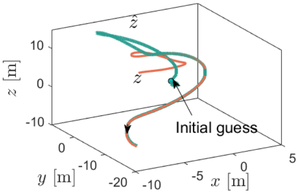
- Obtain the linear regression model $y_{\mathtt{N}} = \psi^\top(t)\theta$.

# Simulation Results

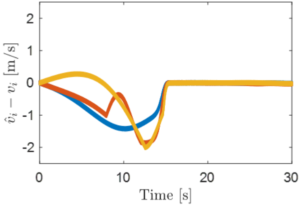
$\psi \in$ IE $\implies$ Global Convergent Range Observer.



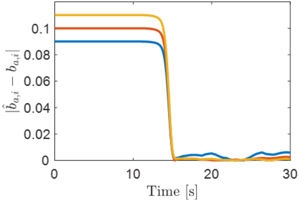

(a) Range estimate $\hat{r}$




(b) Position estimate $\hat{z}$




(c) Velocity estimation error



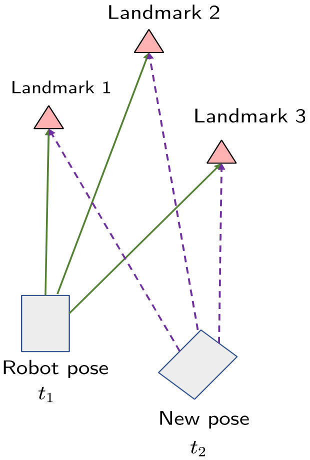

(d) Bias estimation error

# PEBO-SLAM Observer[9]

- A problem formulation:

  *"If we leave a robot in an unknown location and unknown environment can the robot make a satisfactory map while simultaneously being able to find its pose in that map?"*

- Two *concurrent* tasks ("chicken-or-egg")
  - Localisation: inferring pose (position + orientation) given a map
  - Mapping: inferring a map given a location.




[9] Yi, et al, IEEE TAC'2025 PEBO-SLAM: Observer design for visual inertial SLAM with convergence guarantees.

# Dynamical Model (3-dimensional)

System dynamics

$$\dot{\chi} = f(\chi, u)$$

- Kinematics in $\{\mathcal{I}\}$

$$\begin{cases} \dot{R} = R\Omega_\times, & R \in SO(3) \\ \dot{x} = Rv, & x \in \mathbb{R}^3 \end{cases}$$

with $(v, \Omega)$ translational and rotational velocities

- Landmark coordinates

$$\dot{z}_i = 0, \quad z_i \in \mathbb{R}^3, \; i \in \mathcal{N} \subset \mathbb{N}$$

Output function (different sensors)

$$\mathbf{y} = h(\chi) + \underbrace{b_y + \mu_y}_{\text{noise \& bias}}$$

- Range + bearing ($z_i$ in $\{\mathcal{B}\}$) [Radar/stereo cameras]

$$y_i = R^\top (z_i - x)$$

- Bearing [monocular camera]

$$y_i = R^\top \frac{z_i - x}{|z_i - x|}$$

- Range [sonar detect beacons]

$$y_i = |z_i - x|$$

# Is the model observerable/detectable?

- No, neither!
- Why? The **ambiguity of the origin** of the inertial frame.
- How can we do?
  1) Know the absolute coordinates of three landmarks.
  2) Choose the initial pose $X(0)$ (anchor) by the user.
  3) Use the notion of quotient manifold (theoretical analysis) [10]
- Then, the observability depends on
  1) Sensors (output function)
  2) Robot trajectory, i.e. uniform complete observability (UCO) or persistency of excitation (PE)

[10] For example, (Zlotnik & Forbes, TAC'2018)

# Generation of Linear Regressors

- Model: $\dot{R} = R\Omega_\times, \quad \dot{x} = Rv$
- Output: $y_i = R^\top \frac{z_i - x}{|z_i - x|}, \ i \in \mathcal{N}$

- Proposed dynamic extension

$$\dot{Q} = Q\Omega_\times, \quad \dot{\xi} = Qv.$$

- Invariant foliation under distance $E(X_1, X_2) = X_1 X_2^{-1}$ on $SE(3)$

**Lemma**

There exists a constant matrix $X_c \in SE(3)$ s.t.

$$\begin{bmatrix} Q & \xi \\ 0 & 1 \end{bmatrix} \equiv X_c \begin{bmatrix} R & x \\ 0 & 1 \end{bmatrix}, \quad t \geq 0,$$

with $X_c = \begin{bmatrix} Q(0)R(0)^\top & \xi(0) - Q(0)R(0)^\top x(0) \\ 0 & 1 \end{bmatrix}$.

# Generation of Linear Regressors (cont'd)

- Fact: almost all the exisiting works study the output in the inertial frame. (linearisation, or exploiting nonlinearity)
- Can we formulate it as an **on-line** least squares problem?

Lemma: We define the landmark coordinates in $\{\mathcal{V}\}$ as $z_i^v = \xi + QR^\top(z_i - x)$, and the associated bearing $y_i^v := Q^\top \frac{z_i^v - \xi}{|z_i^v - \xi|}$ . Then, we have the relation

$$z_i^v = \xi_c + Q_c z_i, \quad (z_i^v \text{ are constant})$$

and $y_i^v$ are measurable with $y_i^v(t) = y_i(t)$, for all $t \geq 0$.

Obtain linear regressors

$$\Pi_{Q(t)y_i(t)}[z_i^v - \xi(t)] = 0. \tag{44}$$

- Transforming observer design into estimating $X_c$ and $z_i^v$

## Proposition (Landmark Observer under Interval Excitation)

The landmark observer in $\{\mathcal{V}\}$

$$\begin{cases} \dot{\theta}_i = \Delta_i(t)\Big(Y_i - \Delta_i(t)\theta_i\Big) \\ \dot{\omega}_i = -\Delta_i^2(t)\omega_i, \quad \omega_i(0) = 1 \\ \dot{\hat{z}}_i^v = \gamma_i \Delta_i^e(t)\Big[Y_i(t) + k_{\mathtt{I}}^i\Big(\theta_i(t) - \omega_i(t)\theta_i(0)\Big) - \Delta_i^e(t)\hat{z}_i^v\Big] \end{cases} \tag{45}$$

with $\Delta_i^e := \Delta_i + k_{\mathtt{I}}^i(1-\omega_i)$, $\gamma_i > 0$ and $k_{\mathtt{I}}^i > 0$, guarantees

1) (Internal stability) All the internal states are bounded.
2) (Element-wise monotonicity) $\forall t_a \geq t_b \geq 0$,

$$|\hat{z}_{i,j}^v(t_a) - z_{i,j}^v| \leq |\hat{z}_{i,j}^v(t_b) - z_{i,j}^v|.$$

3) (GES of $\tilde{z}_i^v$ under IE) $\Pi_{Q(t)y_i(t)}$ is IE $\implies$ GES

# Pose Observer

## Proposition

Under the IE and non-collinear assumptions. The pose observer

$$\begin{cases} \dot{\bar{z}}_j = \rho_j \bar{\phi}_j(t)\big(\bar{y}_j(t) - \bar{\phi}_j(t)^\top \bar{z}_j\big), \quad j \in \{1, \dots, n_\ell\} \\ \dot{\hat{Q}}_c = -(w_{\texttt{vis}})_\times \hat{Q}_c \\ \dot{\hat{x}} = \hat{R}v(t) + \sum_{j=1}^{n_\ell} \sigma_j\big(\bar{z}_j - \hat{x} - \hat{Q}_c^\top(\hat{z}_j^v - \xi)\big) \\ w_{\texttt{vis}} = \sum_{j=1}^{n_\ell - 1} k_j \hat{r}_j^v \times \left(\hat{Q}_c \bar{r}_j\right), \quad \bar{r}_j = \bar{z}_{j+1} - \bar{z}_j \\ \hat{R} = \hat{Q}_c^\top Q \end{cases} \tag{46}$$

with parameters $\rho_j, k_j, \sigma_j > 0$, $\bar{\phi}_j^\top(t) := \int_0^{\texttt{min}(t,T_\star)} \Pi_{Q(s)y_j(s)} ds \cdot Q(0)R_\star^\top$ and $\bar{y}_j(t) := \int_0^{\texttt{min}(t,T_\star)} \Pi_{Q(s)y_j(s)}\Big(\xi(s) - \xi(0) + Q(0)R_\star^\top x_\star\Big)ds$ and

$$\hat{z}_i = \hat{Q}_c^\top(\hat{z}_i^v - \xi) - \hat{x}, \quad i \in \{1, \dots, n\} \tag{47}$$

achieve the task almost globally.

# PEBO vs Preintegration

- For the dynamics of mobile robots

$$\dot{R} = R\omega_\times, \; {}^{\mathtt{I}}\dot{v} = {}^{\mathtt{I}}a, \; {}^{\mathtt{I}}\dot{x} = {}^{\mathtt{I}}v$$

  the "standard integration" (Picard) is

$$R(t_2) = R(t_1) + \int_{t_1}^{t_2} R(s)[{}^{\mathtt{B}}\bar{\omega}(s) - b_\omega]_\times ds$$

$${}^{\mathtt{I}}v(t_2) = {}^{\mathtt{I}}v(t_1) + \int_{t_1}^{t_2} R(s)[{}^{\mathtt{B}}\bar{a}(s) - b_a]_\times ds + \Delta_t g$$

$$x(t_2) = x(t_1) + \Delta_t {}^{\mathtt{I}}v(t_1) + \frac{1}{2}\Delta_t^2 g + \iint_{t_1}^{t_2} R(s)[{}^{\mathtt{B}}\bar{a}(s) - b_a]ds^2$$

- Issues:
  - Strong nonlinearity
  - Asynchronous sampling: IMU (200 Hz), expensive computation.

Coupling between the IMU integral and the initial condition (Recalculate for the next instance in optimization-based estimation approaches)

- IMU Preintegration (Lupton & Sukkarieh, TRO'11)
  - Widely popular now in the robotics community
  - Change the "integration order"
  - Benefits: 1) Preintegration and IC are separated; 2) Linearity

$$
\begin{aligned}
R(t_{k+1}) &= R(t_k)\Delta R_{t_k}^{t_{+1}} \\
{}^{\mathtt{I}}v(t_k+1) &= {}^{\mathtt{I}}v(t_k) + R(t_k)\Delta v_{t_k}^{t_k+1} + \Delta_t g \\
p(t_{k+1}) &= p(t_k) + \Delta_t {}^{\mathtt{I}}v(t_k) + \frac{1}{2}\Delta_t^2 g + R(t_k)\Delta p_{t_k}^{t_{k+1}}
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta R_{t_1}^{t} &\approx \mathsf{Exp}\left(\int_{t_1}^{t}({}^{\mathtt{B}}\bar{\omega}(s) - b_\omega)ds\right) \\
\Delta v_{t_k}^{t} &= \int_{t_k}^{t} \Delta R_{t_k}^{s}({}^{\mathtt{B}}\bar{a}(s) - b_a)ds \\
\Delta p_{t_k}^{t} &= \iint_{t_k}^{t} \Delta R_{t_k}^{s}({}^{\mathtt{B}}\bar{a}(s) - b_a)ds^2.
\end{aligned}
$$

Preintegration is closely connected to PEBO.

# PEBO vs Preintegration: Euclidean Space [11]

**Proposition.** (Barrau & Bonnabel, SCL'19) Consider the LTV system

$$\dot{x} = A_t x + B_t u, \quad y = C_t x + D_t u. \tag{48}$$

Given two instants $t_k < t_{k+1}$, there exist a matrix $F_k$ and a vector $v_k$ s.t.

$$x(t_{k+1}) = F_k x(t_k) + v_k, \quad \forall x(t_k). \tag{49}$$

- Preintegration signals: $F_k = F(t_{k+1}^-), \quad v_k = v(t_{k+1}^-)$

$$\left.\begin{aligned} \dot{F} &= A_t F, \quad F(t_k^-) = I_n \\ \dot{v} &= A_t v + B_t u, \quad v(t_k^+) = 0_n. \end{aligned}\right\} \text{ Preintegration}$$

- $F(t)$ is the fundamental matrix in a moving horizon.
- PEBO

$$\begin{aligned} \dot{x} &= A_t \xi + B_t u, \quad \xi(t_0) = \xi_0 \\ \dot{\Omega} &= A_t \Omega, \quad \Omega(t_0) = I_n \\ \hat{\theta} &= \operatorname*{argmin}_{\theta \in \mathbb{R}^n} \sum_{k=0}^{N} \gamma_k \left| Y(t_k) - C(t_k)\Omega(t_k)\theta \right|^2 \end{aligned} \tag{50}$$

[11] Yi & Manchester, SCL'2024, On IMU preintegration: A nonlinear observer viewpoint and its application.

# The PEBO Viewpoint to Preintegration

Proposition. State estimation using preintegration **exactly coincides** with the PEBO with zero initial condition in the following sense:

1. The preintegration signal $F$ and fundamental matrix $\Omega$ satisfy

$$\begin{aligned} \Omega(t_k) &= \prod_{i=0}^{k-1} F_i := F_{k-1} \dots F_0, \quad \forall k \in \mathbb{N} \\ \Omega(t) &= F(t)\Omega(t_k), \quad t \in (t_k, t_{k+1}). \end{aligned} \tag{51}$$

2. The preintegration signal $v$ and the dynamic extension variable $\xi$ verify

$$v_t = \xi_t - \Omega_t \Omega(t_k)^{-1} \xi(t_k), \quad t \in (t_k, t_{k+1}). \tag{52}$$

3. The PEBO estimate equals to the one from preintegration, i.e., $\hat{\mathbf{x}}_{\mathtt{PEBO}} = \hat{\mathbf{x}}_{\mathtt{PI}}$.

# The PEBO Viewpoint to Preintegration

- Preintegration can be viewed as the implementation of PEBO in moving horizons.
- With uncertainties, the estimate would be (slightly) different.
- Similar results holds true for IMU preintegration on SO(3)$\times\mathbb{R}^n$ (more technical)
- IMU preintegration uses the inertial coordinate, and PEBO in the body-fixed coordinate (linear parameterisation to bias)
- Improve long-term performance and robustness
- Applications of our findings:
  1. Hybrid sampled-data observer
  2. PEBO for stochastic systems
  3. IMU preintegration using body-fixed frame

# Outline

# From PEBO to Gradient Observers

In some models, if the we have

- the derivatives of the unknown $x$, i.e.

$$\dot{x} = f(u, y)$$

- the regression model of $x$

$$h(x, t) = 0.$$

We may design a gradient observer directly

$$\dot{\hat{x}} = f(u, y) - \gamma \nabla_{\hat{x}} J(\hat{x}, t)$$

with $J(\hat{x}, t)$ a cost function related to the regressor.

The minima of $J(\hat{x}, t)$ include the (time-varying) solution to the regressor $h(x, t) = 0$.

Dynamical Model:

$$\dot{\lambda} = -Ri + v$$

Measured Output:

$$i = \begin{bmatrix} L_s + \frac{L_0}{2}\cos 2\theta & \frac{L_0}{2}\sin 2\theta \\ \frac{L_0}{2}\sin 2\theta & L_s - \frac{L_0}{2}\cos 2\theta \end{bmatrix}^{-1} \left( \lambda - \psi_m \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \right)$$

with $\theta = \arccos(\lambda_2, \lambda_1)$.

# Generating of Perturbed Linear Regressor via Filtering

$$\boxed{y = \Phi^\top \mathbf{x} + d + \epsilon_t,} \tag{53}$$

where the active flux $\mathbf{x} := \lambda - L_q \mathbf{i}$, and the *measurable* signals $y(t) \in \mathbb{R}^2$ and $\Phi(t) \in \mathbb{R}^2$ are given as

$$y \ := \ L_0 H_2[\mathbf{i}]^\top \Omega_1 + \frac{1}{\alpha}|\Omega_1|^2 + \frac{1}{\alpha} H_2[\Omega_2^\top \Omega_1], \quad \Phi \ := \ \Omega_1 + \Omega_2. \tag{54}$$

with the signals $\Omega_1(t) \in \mathbb{R}^2$ and $\Omega_2(t) \in \mathbb{R}^2$ defined as

$$\Omega_1 := H_2[\mathbf{v} - R\mathbf{i} - L_q p \mathbf{i}], \quad \Omega_2 := \Omega_1 - L_0 H_1[\mathbf{i}],$$

with the filters $H_1(p) := \frac{\alpha p}{p+\alpha}$, $H_2(p) := \frac{\alpha}{p+\alpha}$. The (unknown) perturbing signal $d$ is given by

$$d := -\ell H_1 \left[ \mathbf{i}^\top \frac{\mathbf{x}}{|\mathbf{x}|} \right].$$

# Gradient-Like Flux Observer[12]

$$\dot{\hat{\lambda}} = \underbrace{\mathbf{v} - R\mathbf{i}}_{\text{flux dynamics}} + \underbrace{\gamma\Phi\left(y - \Phi^\top \hat{\mathbf{x}} + \ell\frac{\alpha p}{p+\alpha}\left[\mathbf{i}^\top \sigma(\hat{\mathbf{x}})\right]\right)}_{\text{gradient-like}}$$

$$\hat{\mathbf{x}} = \hat{\lambda} - L_q\mathbf{i}$$

$$\hat{\theta} = \mathrm{atan2}(\hat{\mathbf{x}}_2, \hat{\mathbf{x}}_1),$$

If $\Phi$ is PE and $\gamma > 0$ sufficiently small, we achieve exponential stability.

- Small $\gamma > 0$ limits the performance.
- Hint: small $\gamma$ approx. makes the convergence rate $\propto \gamma$
  (Anderson, Bitmead, Johnson, Kokotovic, Kosut, Mareels, Praly and Riedle, Stability of Adaptive Systems: Passivity and Averaging Analysis, 1986.)

[12] Ortega, Yi, et al., Automatica 2021, A globally exponentially stable position observer for interior permanent magnet synchronous motors.

# Kreisselmeier's Gradient Observer[13]

Kreisselmeier's extension:

$$y = \phi(t)^\top \theta \quad \xrightarrow[\text{LTV filtering}]{} \quad Y = \Psi(t)\theta$$

to make $\Psi(t) \succeq \alpha I$ after $t_\star$.

For state observation, $\mathbf{x}(t)$ is time-varying and has its dynamics. How can we utilize the idea?

$$y = \Phi^\top \mathbf{x}(t) + d + \epsilon_t$$

**Our Solution:** Make compensation in observer dynamics to obtain forward invariant manifold

$$\mathcal{M} := \{(Y, Q, \xi, \tilde{\mathbf{x}}) : Y = Q\tilde{\mathbf{x}} + \xi\},$$

[13] Yi et al., Automatica'2025, A high performance globally exponentially convergent sensorless observer for the IPMSM: Theoretical and experimental results

# Kreisselmeier's Gradient Observer

*Flux and position observer*

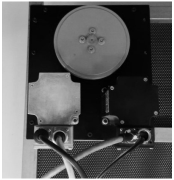
$$
\left.\begin{aligned}
\dot{Q} &= -a\big(Q - \Phi\Phi^\top\big) \\
\dot{Y} &= -a\big(Y - \Phi e\big) + QE \\
E &= -\gamma Y
\end{aligned}\right\} \text{(Kreisselmeier's Extension)}
$$
$$
\left.\begin{aligned}
\dot{\hat{\lambda}} &= \mathbf{v} - R\mathbf{i} + E \\
\hat{\mathbf{x}} &= \hat{\lambda} - L_q\mathbf{i} \\
\hat{\theta} &= \mathrm{atan2}(\hat{\mathbf{x}}_2, \hat{\mathbf{x}}_1),
\end{aligned}\right\} \text{(Flux-position estimate)} \tag{55}
$$

with

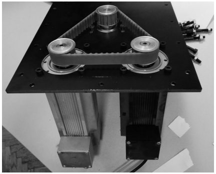
$$
\hat{d} = -\ell H_1[\mathbf{i}^\top \sigma(\hat{\mathbf{x}})],\ e := \Phi^\top \hat{\mathbf{x}} + \hat{d} - y. \tag{56}
$$

No need to use low adaptation gain: High estimation performance!

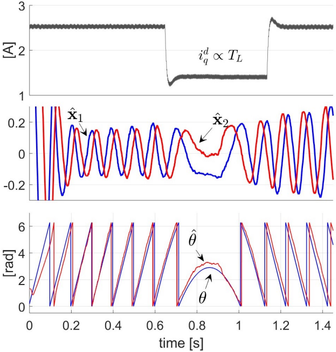
# Experimental Results



(a) Experimental testing setup



(b) Shafts connection of two motors




(c) Angle and active flux estimates with speed reversal under pulse load condition

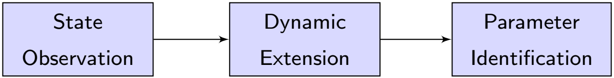
Beyond PEBO and Gradient Observer:

A more general recursive optimization-based observer framework?

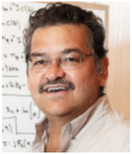
# Outline

# Summary

- A new observer design framework for nonlinear systems
- Translate into online parameter identification
- Effective in solving various practical problems, particularly in robotics
- Future works:
  1. Enhance robustness
  2. Advanced online optimization techniques
  3. General model class
  4. Stochastic perspective



# Acknowledgement



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# THANKS

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